

# Semiclassical Trace Formulae and Eigenvalue Statistics in Quantum Chaos<sup>1</sup>

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## Abstract

A detailed discussion of semiclassical trace formulae is presented and it is demonstrated how a regularized trace formula can be derived while dealing only with finite and convergent expressions. Furthermore, several applications of trace formula techniques to quantum chaos are reviewed. Then local spectral statistics, measuring correlations among finitely many eigenvalues, are reviewed and a detailed semiclassical analysis of the number variance is given. Thereafter the transition to global spectral statistics, taking correlations among infinitely many quantum energies into account, is discussed. It is emphasized that the resulting limit distributions depend on the way one passes to the global scale. A conjecture on the distribution of the fluctuations of the spectral staircase is explained in this general context and evidence supporting the conjecture is discussed.

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<sup>1</sup> Lectures held at the 3rd International Summer School/Conference *Let's face chaos through nonlinear dynamics* at the University of Maribor, Slovenia, 24 June – 5 July 1996

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# Introduction

In a dynamical system deterministic chaos manifests itself in an effective unpredictability of the dynamics for large times. Different degrees of a chaotic behaviour can be observed by identifying the properties of ergodicity, mixing, positive dynamical entropies, etc. [1]. All of these concepts require to investigate the limit  $t \rightarrow \infty$  of the dynamics. In the following we will restrict attention to the –important– cases of autonomous, bound Hamiltonian dynamics. Nevertheless, one can still find examples for all types of different chaotic behaviour. However, upon quantizing such classical dynamical systems chaos in the above sense disappears. In a loose manner one could explain this observation by the regularizing effect the quantum mechanical uncertainty principle has on an irregular classical dynamics. More precisely, the discrete spectrum of the quantum Hamiltonian  $\hat{H}$  enforces the time evolution in quantum mechanics to be almost periodic. In contrast, an almost periodic classical time evolution is only observed for integrable systems. If one prepares a quantum system at time  $t_0 = 0$  in a state  $\psi_0$ , the solution  $\psi(t)$  of the Schrödinger equation at time  $t > 0$  reads

$$\psi(t) = \sum_n c_n \varphi_n e^{-\frac{i}{\hbar} E_n t} , \quad (0.1)$$

where  $\varphi_n$  and  $E_n$  denote the eigenvectors and eigenvalues of the quantum Hamiltonian, respectively:  $\hat{H}\varphi_n = E_n\varphi_n$ . As  $t \rightarrow \infty$  the state described by (0.1) fluctuates in a possibly very wild manner, but cannot approach zero. More importantly, the same observation holds for the correlation function

$$\langle \psi_0, \psi(t) \rangle = \sum_n |c_n|^2 e^{-\frac{i}{\hbar} E_n t} , \quad (0.2)$$

which has the same structure as (0.1). Thus no mixing can occur, nor can any of the other features conventionally characterising chaos be observed. However, in the classical limit, which is formally obtained by setting Planck's constant  $\hbar = 0$ , one recovers integrable as well as chaotic dynamics and all possible situations in between. The precise behaviour of the classical system is dictated by the Hamilton function, which generates the classical time evolution. The latter can produce correlation functions that range from quasi-periodic ones to exponentially decaying ones. The reason for this seemingly paradoxical finding is that the two limits involved,  $\hbar \rightarrow 0$  and  $t \rightarrow \infty$ , do not commute.

The field of quantum chaos [2, 3] now is concerned with a search for fingerprints the chaotic behaviour of its classical limit leaves on a quantum system. Since due to (0.1) the spectrum and the eigenfunctions of the quantum Hamiltonian completely determine the time evolution of a quantum system, it seems natural to investigate statistical properties of eigenvalues and eigenfunctions. The goal of the following lectures now is to explain in some detail an approach to eigenvalue statistics employing trace formulae. In the first part we introduce semiclassical trace formulae and discuss some of their applications, including a discussion of the use of zeta functions. The second part then is devoted to an application of trace formulae to eigenvalue statistics. A certain aspect of the latter, namely the approach to the global distribution of eigenvalues, will be the central theme of the third part.

## 1 Introduction to Trace Formulae and Zeta Functions

The basic idea behind semiclassical trace formulae is to take the trace of the quantum mechanical time evolution operator, thereby loosing all information on eigenfunctions, and to exploit the Fourier duality between the time dependent and the energy dependent picture. In order to deal with well behaved quantities, in the course of the required manipulations one has to employ a certain regularization procedure. Finally then, one can express certain sums over the spectrum of a quantum Hamiltonian by sums over periodic orbits of the corresponding classical dynamics. In certain cases one is furthermore able to determine explicitly the leading order contributions as  $\hbar \rightarrow 0$ .

To get an idea of the kind of quantum systems to which one can apply the following procedure we now list three of the more prominent types of examples:

1. Schrödinger operators of the form  $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(x)$ , where here  $\Delta$  denotes the Laplacian for  $\mathbb{R}^d$  and  $V(x)$  is a suitable potential. If  $V(x) \rightarrow \infty$  for  $|x| \rightarrow \infty$  sufficiently fast, it is ensured that  $\hat{H}$  has a discrete spectrum. The corresponding classical dynamics on the phase space  $\mathbb{R}^d \times \mathbb{R}^d$  is generated by the Hamilton function  $H(p, x) = \frac{p^2}{2m} + V(x)$ .
2. Quantum billiards: Let  $D \subset \mathbb{R}^d$  be a compact domain with a sufficiently well behaved boundary  $\partial D$ . Then  $\hat{H} = -\frac{\hbar^2}{2m}\Delta$ , where now  $\Delta$  denotes either the Dirichlet- or the Neumann-Laplacian for  $D$ . The corresponding classical dynamics on the phase space  $D \times S^{d-1}$  is that of a free motion inside  $D$  with elastic reflections from  $\partial D$ .
3. Let  $M$  be a compact Riemannian manifold of dimension  $d$ . Then  $\hat{H} = -\frac{\hbar^2}{2m}\Delta$ , where now  $\Delta$  is the Laplace-Beltrami operator for the Riemannian metric on  $M$ . The classical phase space is provided by the unit cotangent bundle over  $M$ , and the classical dynamics is that of the geodesic flow, i.e., the free motion along geodesics.

When it comes to explicit formulae, we will in the following primarily adhere to case 1. However, possibly after some more or less obvious modifications, the results carry over to the other cases as well.

In all of the above cases the spectrum of  $\hat{H}$  is discrete and bounded from below. One can therefore add a suitable constant to  $\hat{H}$  in order to render the quantum Hamiltonian non-negative. Thus

$$0 \leq E_1 \leq E_2 \leq E_3 \leq \dots \quad (1.1)$$

provides the list of quantum energies in which we count each eigenvalue with its respective multiplicity.

## 1.1 Semiclassical Trace Formulae

The crudest semiclassical approximation to quantum mechanics can be summarized in the following rule: Each eigenstate of the quantum Hamiltonian corresponds to a cell of volume  $(2\pi\hbar)^d$  in the classical phase space. Brought in a more mathematical form this statement yields the leading semiclassical asymptotics for the spectral staircase,

$$N(E) := \#\{n; E_n \leq E\} \sim \frac{1}{(2\pi\hbar)^d} \int \int \Theta(E - H(p, x)) dp dx, \quad \hbar \rightarrow 0. \quad (1.2)$$

For the density of states this means

$$d(E) = \frac{d}{dE} N(E) = \sum_n \delta(E - E_n) \sim \frac{1}{(2\pi\hbar)^d} \int \int \delta(E - H(p, x)) dp dx, \quad \hbar \rightarrow 0. \quad (1.3)$$

The ultimate goal of a semiclassical trace formula now is to provide correction terms to the asymptotics on the r.h.s. of (1.2) and (1.3). But since the spectral density  $d(E)$  is a highly singular object, with distributional singularities at the eigenvalues  $E_n$ , one has to employ some smoothing procedure.

To start with let us consider the time evolution operator  $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$ , so that

$$\psi(t, x) = \left( \hat{U}(t) \psi_0 \right) (x) = \int K(x, y; t) \psi_0(y) dy \quad (1.4)$$

is the solution of the Schrödinger equation with initial condition  $\psi(0, x) = \psi_0(x)$ . The distributional kernel  $K(x, y; t)$  then obeys the Schrödinger equation

$$\left( i\hbar \frac{\partial}{\partial t} - \hat{H}_x \right) K(x, y; t) = i\hbar \delta(x - y) \delta(t), \quad (1.5)$$

with initial condition

$$\lim_{t \rightarrow 0} K(x, y; t) = \delta(x - y) . \quad (1.6)$$

It can for  $t \geq 0$  be expanded in an orthonormal basis of eigenfunctions  $\{\varphi_n(x)\}$  of  $\hat{H}$ ,

$$K(x, y; t) = \sum_n \varphi_n(x) \overline{\varphi_n(y)} e^{-\frac{i}{\hbar} E_n t} , \quad (1.7)$$

so that the formal trace of the time evolution operator is given by  $\text{Tr } \hat{U}(t) = \sum e^{-\frac{i}{\hbar} E_n t}$ . This object does in general not exist as a smooth function of  $t$  because it has singularities, the most obvious one being at  $t = 0$ . In order to regularize the trace consider a smooth function  $\rho(t)$  with a compact support and define the bounded operator

$$\hat{U}[\rho] := \int_{-\infty}^{+\infty} \rho(t) \hat{U}(t) dt , \quad (1.8)$$

which has a finite trace, since

$$\text{Tr } \hat{U}[\rho] = \int dx \int_{-\infty}^{+\infty} dt \rho(t) K(x, x; t) = \sum_n \int_{-\infty}^{+\infty} \rho(t) e^{-\frac{i}{\hbar} E_n t} dt = \sum_n \hat{\rho} \left( -\frac{E_n}{\hbar} \right) . \quad (1.9)$$

Here  $\hat{\rho}(E)$  denotes the Fourier transform of the function  $\rho(t)$ . Since  $\rho(t)$  was required to be smooth and compactly supported, its Fourier transform is a Schwartz-class test function, i.e.,  $\hat{\rho}(E)$  and all of its derivatives decrease faster than any power. Once then the eigenvalues  $E_n$  behave as  $E_n \sim \text{const. } n^\alpha$ ,  $n \rightarrow \infty$ , for some constant  $\alpha$ , the sum on the r.h.s. of (1.9) is finite. Therefore, the map  $\rho \mapsto \text{Tr } \hat{U}[\rho]$  is a tempered distribution.

We now choose the test function  $\rho(t) e^{\frac{i}{\hbar} E t}$  and hence observe

$$\text{Tr } \hat{U} \left[ \rho(t) e^{\frac{i}{\hbar} E t} \right] = \int dx \int_{-\infty}^{+\infty} dt \rho(t) e^{\frac{i}{\hbar} E t} K(x, x; t) = \sum_n \hat{\rho} \left( \frac{E - E_n}{\hbar} \right) . \quad (1.10)$$

For convenience we rename the test functions in such a way that  $\varphi(E) := \hat{\rho}(-E)$ , and thus  $\hat{\varphi}(t) = 2\pi\rho(t)$ . Then (1.10) reads

$$\sum_n \varphi \left( \frac{E_n - E}{\hbar} \right) = \frac{1}{2\pi} \int dx \int_{-\infty}^{+\infty} dt \hat{\varphi}(t) e^{\frac{i}{\hbar} E t} K(x, x; t) . \quad (1.11)$$

Here  $\varphi(E)$  and  $\hat{\varphi}(t)$  are required to be smooth functions, and  $\hat{\varphi}(t)$  shall have a compact support. The strategy now is to find a semiclassical approximation for the r.h.s., which then yields a semiclassical approximation for the spectral density  $d(E)$ . To see this one notices that

$$\hbar d(E + \hbar\varepsilon) = \sum_n \hbar \delta(E + \hbar\varepsilon - E_n) = \sum_n \delta \left( \varepsilon - \frac{E_n - E}{\hbar} \right) , \quad (1.12)$$

so that upon evaluating (1.12) on the test function  $\varphi(\varepsilon)$  one obtains

$$\langle \hbar d(E + \hbar\varepsilon), \varphi(\varepsilon) \rangle = \sum_n \varphi \left( \frac{E_n - E}{\hbar} \right) . \quad (1.13)$$

Some of the general features of a semiclassical approximation for the r.h.s. of (1.11) can be revealed once one chooses an ansatz for the kernel of the time evolution operator. Let us therefore assume that

$$K(x, y; t) = \frac{1}{(2\pi\hbar)^d} \int e^{\frac{i}{\hbar} \phi(x, y, t; q)} a_\hbar(x, y, t; q) dq , \quad (1.14)$$

where  $q \in \mathbb{R}^d$  is some auxiliary variable. In order to achieve the initial condition (1.6) for  $K(x, y; t)$  at  $t = 0$  one demands that

$$\phi(x, y, 0; q) = (x - y)q \quad \text{and} \quad a_{\hbar}(x, y, 0; q) = 1 . \quad (1.15)$$

Inserting the ansatz (1.14) into the Schrödinger equation (1.5) for  $t > 0$  then yields an equation for the phase  $\phi$  and the amplitude  $a_{\hbar}$ . Assuming that the amplitude allows for a formal expansion in powers of  $\hbar$ ,

$$a_{\hbar}(x, y, t; q) = \sum_{k \geq 0} (i\hbar)^k a_k(x, y, t; q) , \quad (1.16)$$

one compares like powers of  $\hbar$ . As a result, one obtains to lowest order in  $\hbar$  the Hamilton-Jacobi equation [4, 5]

$$\frac{\partial \phi(x, y, t; q)}{\partial t} + H(\nabla_x \phi(x, y, t; q), x) = 0 \quad (1.17)$$

for the phase. The higher orders yield transport equations for the  $k$ -th order coefficients  $a_k$ . In principle, starting with the Hamilton-Jacobi equation (1.17) one could now successively solve these equations order by order and thus construct the time evolution kernel (1.14) to any desired precision in  $\hbar$ . Our ambition in what follows, however, will be restricted to obtain the leading order semiclassical asymptotics.

A solution  $\phi(x, y, t; q)$  of the Hamilton-Jacobi equation (1.17) with initial condition (1.15) turns out to be closely related to a generating function for a canonical transformation in phase space that describes the classical dynamics backwards in time. One introduces  $S(x, q, t) := \phi(x, y, t; q) + yq$ , which obviously solves the Hamilton-Jacobi equation (1.17) with initial condition  $S(x, q, 0) = xq$ . Then, if  $(q, y)$  is an initial condition for the solution of the classical equations of motion that reaches the phase space point  $(p, x)$  at time  $t$  with energy  $E$ , the function  $S(x, q, t)$  generates the canonical transformation  $(p, x) \mapsto (q, y)$ . The formalism of canonical transformations and generating functions [4, 5] hence provides one with necessary conditions to be imposed on the phase appearing in the ansatz (1.14),

$$\begin{aligned} \nabla_x \phi(x, y, t; q) &= p & \text{and} & & \nabla_y \phi(x, y, t; q) &= -q , \\ \frac{\partial}{\partial t} \phi(x, y, t; q) &= -E & \text{and} & & \nabla_q \phi(x, y, t; q) &= 0 , \end{aligned} \quad (1.18)$$

whenever  $(q, y)$  is the initial condition for a classical trajectory with energy  $E$  in phase space that passes through  $(p, x)$  at time  $t$ .

We are now in a position to insert the ansatz (1.14) into (1.11),

$$\sum_n \varphi \left( \frac{E_n - E}{\hbar} \right) = \frac{1}{2\pi(2\pi\hbar)^d} \int dx \int dt \int dq \, \hat{\varphi}(t) e^{\frac{i}{\hbar}[\phi(x, x, t; q) + Et]} \sum_{k \geq 0} (i\hbar)^k a_k(x, x, t; q) . \quad (1.19)$$

Inspecting the r.h.s. of (1.19) one realizes that an application of the method of stationary phase [5, 6] to the multiple integral allows to obtain the asymptotic behaviour of this expression as  $\hbar \rightarrow 0$ . To this end one has to identify the points  $(x, t, q)$  at which the total phase  $\phi(x, x, t; q) + Et$  becomes stationary, i.e., where

$$\begin{aligned} 0 &= [\nabla_x \phi(x, y, t; q) + \nabla_y \phi(x, y, t; q)]_{y=x} , \\ 0 &= \frac{\partial}{\partial t} \phi(x, x, t; q) + E , \\ 0 &= \nabla_q \phi(x, x, t; q) . \end{aligned} \quad (1.20)$$

A comparison with (1.18) now immediately reveals that (1.20) picks out those triples  $(x, t, q)$  such that  $(q, x)$  is a point in phase space that is connected to itself by a solution of the classical equations

of motion with energy  $E$  and initial condition  $(q, y) = (q, x)$  at  $t_0 = 0$ , and  $(p, x) = (q, x)$  at  $t$ . Hence, exactly when  $t$  is a period of some classical periodic orbit with energy  $E$ , and  $(q, x)$  is a point in phase space on such an orbit, then  $(x, t, q)$  is a stationary point for the total phase in (1.19). On the contrary, if  $t$  does not correspond to any period of a periodic orbit, no stationary point in the integral over  $(x, q)$  occurs. If one therefore chooses the test function  $\hat{\varphi}(t)$  in such a way that it vanishes on all periods of classical periodic orbits with energy  $E$ , the method of stationary phase [5, 6] yields the estimate  $O(\hbar^\infty)$ , as  $\hbar \rightarrow 0$  in (1.19).

Apart from terms of  $O(\hbar^\infty)$ , hence all relevant contributions to (1.19) come from periodic orbits of the classical dynamics. In this context a distinguished role is played by the time  $t = 0$ , since then the whole hypersurface  $\Omega_E := \{(p, x); H(p, x) = E\}$  of energy  $E$  in phase space is one huge manifold of stationary points. The leading order as  $\hbar \rightarrow 0$  of its contribution to (1.19) can, however, be explicitly calculated. To this end let  $\chi(t)$  be a smooth function with  $\chi(t) = 1$  in a small neighbourhood of  $t = 0$ , which vanishes outside a somewhat larger neighbourhood of  $t = 0$ . It shall in particular vanish on all periods of non-trivial classical periodic orbits with energy  $E$ . The trivial identity  $1 = \chi(t) + [1 - \chi(t)]$  will then be introduced under the multiple integral in (1.19) such that the contribution

$$\frac{1}{2\pi(2\pi\hbar)^d} \int dx \int dt \int dq \chi(t) \hat{\varphi}(t) e^{\frac{i}{\hbar}[\phi(x, x, t; q) + Et]} \sum_{k \geq 0} (i\hbar)^k a_k(x, x, t; q) \quad (1.21)$$

of the stationary points  $(x, t = 0, q)$  is separated from the further stationary points with  $t > 0$ . If one now introduces polar coordinates for  $q$ ,  $q = \lambda\omega$ ,  $\lambda = |q|$ ,  $|\omega| = 1$ , one can perform the integrals over the variables  $t$  and  $\lambda$  by the method of stationary phase. As a result, (1.21) yields

$$\frac{\text{vol}(\Omega_E)}{(2\pi\hbar)^{d-1}} \frac{\hat{\varphi}(0)}{2\pi} \{1 + O(\hbar)\} , \quad \hbar \rightarrow 0 , \quad (1.22)$$

see [6, ch.12] for a closely related problem. If one exploits the relation (1.13), the expression (1.22) indeed reproduces for the spectral density  $d(E)$  the leading asymptotic term as given in (1.3).

Although the leading asymptotic behaviour has already been determined, the real challenge is to calculate corrections to this. One therefore has to consider the remaining part

$$\frac{1}{2\pi} \int dx \int dt [1 - \chi(t)] \hat{\varphi}(t) e^{\frac{i}{\hbar}Et} K(x, x; t) , \quad (1.23)$$

of (1.11), which due to the above considerations is modulo terms of  $O(\hbar^\infty)$  completely dominated by the contributions of non-trivial classical periodic orbits. Since in (1.23) the test function  $\hat{\varphi}(t)$  is required to be of compact support the time integral indeed extends only over a finite interval. It therefore suffices to know  $K(x, x; t)$  for a certain bounded range of  $t$ -values. This observation turns out to be essential for the further manipulations performed in (1.23) in order to obtain a semiclassical approximation. Namely, the first step in this direction consists of obtaining the leading semiclassical asymptotics for the time evolution kernel. In this context one naturally considers  $\hbar \rightarrow 0$  for fixed  $t$ . An application of the resulting asymptotic expression in (1.23), however, requires to use the latter on the whole range of integration. If it were not for the test function  $\hat{\varphi}(t)$ , this would involve  $t \rightarrow \infty$ . But the two limits  $\hbar \rightarrow 0$  with  $t$  fixed, and  $t \rightarrow \infty$  with  $\hbar$  fixed, do not commute, as already mentioned in the Introduction. In mathematical terms this means that the semiclassical estimate is not uniform in  $t$ . This remark should stress the indispensable role played by the regularization procedure applied in the course of the present discussion.

At least for small  $t$ , the calculation of the leading order semiclassical asymptotics for the time evolution kernel goes back to Pauli [7]. It reads

$$K(x, y; t) = \frac{1}{(2\pi i\hbar)^{\frac{d}{2}}} \sum_{\gamma_{x,y}} \left| \det \left( -\frac{\partial^2 R_{\gamma_{x,y}}}{\partial x_k \partial y_l} \right) \right|^{\frac{1}{2}} e^{\frac{i}{\hbar} R_{\gamma_{x,y}}(x, y; t) - i\frac{\pi}{2} \nu_{\gamma_{x,y}}} \{1 + O(\hbar)\} , \quad (1.24)$$

where the sum extends over all solutions  $\gamma_{x,y}$  of the classical equations of motion with boundary conditions  $\gamma_{x,y}(0) = y$  and  $\gamma_{x,y}(t) = x$ . Furthermore, the quantity  $R_{\gamma_{x,y}}(x, y; t)$  is defined as the integral of the Lagrangian  $L(x, \dot{x}) = \frac{m}{2}\dot{x}^2 - V(x)$  along  $\gamma_{x,y}$ ,

$$R_{\gamma_{x,y}}(x, y; t) := \int_0^t L(\gamma_{x,y}(t'), \dot{\gamma}_{x,y}(t')) dt' . \quad (1.25)$$

Finally,  $\nu_{\gamma_{x,y}}$  denotes the number of points on the trajectory  $\gamma_{x,y}$  which are conjugate to the initial point  $y$ . For small  $t$ ,  $\nu_{\gamma_{x,y}}$  vanishes, and this is the case covered by Pauli. The modification required when passing through conjugate points as  $t$  increases was later provided by Gutzwiller [8].

Inserting the semiclassical approximation (1.24) for the time evolution kernel into (1.23) yields the expression

$$\frac{1}{2\pi (2\pi i\hbar)^{\frac{d}{2}}} \int dx \int dt [1 - \chi(t)] \hat{\varphi}(t) \sum_{\gamma_{x,x}} \left| \det \left( -\frac{\partial^2 R_{\gamma_{x,y}}}{\partial x_k \partial y_l} \right)_{y=x} \right|^{\frac{1}{2}} \cdot e^{\frac{i}{\hbar} [R_{\gamma_{x,x}}(x, x; t) + Et] - i\frac{\pi}{2} \nu_{\gamma_{x,x}}} \{1 + O(\hbar)\} . \quad (1.26)$$

Since in general an explicit evaluation of (1.26) is out of reach one employs the method of stationary phase to obtain the leading semiclassical asymptotics. In order to determine the stationary points of the phase  $R_{\gamma_{x,x}}(x, x; t) + Et$  one has to solve

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} [R_{\gamma_{x,y}}(x, y; t) + Et]_{y=x} , \\ 0 &= \nabla_x R_{\gamma_{x,y}}(x, y; t)|_{y=x} + \nabla_y R_{\gamma_{x,y}}(x, y; t)|_{y=x} . \end{aligned} \quad (1.27)$$

From classical mechanics [4] one recalls the identities

$$\nabla_x R_{\gamma_{x,y}}(x, y; t) = p , \quad \nabla_y R_{\gamma_{x,y}}(x, y; t) = -q , \quad \frac{\partial}{\partial t} R_{\gamma_{x,y}}(x, y; t) = E_{\gamma_{x,y}} , \quad (1.28)$$

where  $q$  and  $p$  denote the momenta along  $\gamma_{x,y}$  at the instants  $t_0 = 0$  and  $t$ , respectively.  $E_{\gamma_{x,y}}$  then is the energy of this trajectory. Thus, (1.27) picks out those closed classical trajectories  $\gamma_{x,x}$  that share identical initial and final momenta,  $q = p$ ; hence these are periodic orbits with energy  $E$ . That way one recovers the previous observation, see (1.19)–(1.20), that all relevant contributions to (1.10) are due to classical periodic orbits of energy  $E$ . However, one now is in a position to calculate the leading semiclassical contribution to (1.11) explicitly.

As a first step, consider the contribution of non-trivial periodic orbits to the regularized Green function,

$$G_{po}^\varphi(x, y; E) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} [1 - \chi(t)] \hat{\varphi}(t) K(x, y; t) e^{\frac{i}{\hbar} Et} dt . \quad (1.29)$$

In (1.26) one therefore disregards the integration over  $x$  and moreover reintroduces  $y \neq x$ . A well known calculation of the integral over  $t$  with the help of the method of stationary phase, see for example [8, 2], then yields

$$G_{po}^\varphi(x, y; E) = \sum_{\gamma_{x,y}} A_{\gamma_{x,y}}^\varphi(x, y; E) e^{\frac{i}{\hbar} S_{\gamma_{x,y}}(x, y; E) - i\frac{\pi}{2} \mu_{\gamma_{x,y}}} \{1 + O(\hbar)\} . \quad (1.30)$$

Here

$$A_{\gamma_{x,y}}^\varphi(x, y; E) := \frac{\hat{\varphi}(T_{\gamma_{x,y}})}{2\pi(2\pi i\hbar)^{\frac{d-1}{2}}} \sqrt{D_{\gamma_{x,y}}(x, y; E)} \quad (1.31)$$

is an amplitude factor attached to each classical trajectory  $\gamma_{x,y}$  connecting  $y$  and  $x$  with fixed energy  $E$ . The time  $T_{\gamma_{x,y}}$  needed for this may depend on the particular trajectory. Moreover,

$$D_{\gamma_{x,y}}(x, y; E) := \left| \frac{\det \left( -\frac{\partial^2 R_{\gamma_{x,y}}}{\partial x_k \partial y_l} \right)}{\frac{\partial^2 R_{\gamma_{x,y}}}{\partial t^2}} \right| = \left| \det \begin{pmatrix} \frac{\partial^2 S_{\gamma_{x,y}}}{\partial x_k \partial y_l} & \frac{\partial^2 S_{\gamma_{x,y}}}{\partial x_k \partial E} \\ \frac{\partial^2 S_{\gamma_{x,y}}}{\partial y_l \partial E} & \frac{\partial^2 S_{\gamma_{x,y}}}{\partial E^2} \end{pmatrix} \right|, \quad (1.32)$$

$$\mu_{\gamma_{x,y}} := \begin{cases} \nu_{\gamma_{x,y}} & , \quad \frac{\partial^2 R_{\gamma_{x,y}}}{\partial t^2} > 0 \\ \nu_{\gamma_{x,y}} + 1 & , \quad \frac{\partial^2 R_{\gamma_{x,y}}}{\partial t^2} < 0 \end{cases}. \quad (1.33)$$

Finally,  $S_{\gamma_{x,y}}(x, y; E) := R_{\gamma_{x,y}}(x, y; T_{\gamma_{x,y}}) + ET_{\gamma_{x,y}}$  can also be expressed as the integral of  $p dx$  along the trajectory  $\gamma_{x,y}$ .

In a second step, the trace of  $G_{po}^\varphi$  will be calculated by integrating (1.30) over the diagonal  $x = y$  with respect to  $x$ . Once again the method of stationary phase will be applied, and to this end the condition of stationarity for the phase,

$$[\nabla_x S_{\gamma_{x,y}}(x, y; E) + \nabla_y S_{\gamma_{x,y}}(x, y; E)]_{x=y} = p - q \stackrel{!}{=} 0, \quad (1.34)$$

yields all points  $x$  on classical periodic orbits with energy  $E$  as solutions. These points are obviously not isolated, as it would be required by the simplest version of the method of stationary phase, since already a single periodic orbit  $\gamma$  constitutes a one dimensional connected manifold of stationary points.

However, in case the set of stationary points  $x$  devides into a series  $M_j$ ,  $j = 1, 2, 3, \dots$ , of connected smooth manifolds of dimensions  $m_j \leq d$  in configuration space, one can introduce local coordinates in suitable neighbourhoods of the  $M_j$ 's such that  $x \mapsto (u, v)$ , where  $u = (u_1, \dots, u_{m_j})$  parametrizes  $M_j$ . The method of stationary phase is then applied to the integral over the transversal coordinates  $v = (v_1, \dots, v_{d-m_j})$ . As a result, one obtains that

$$\text{Tr } G_{po}^\varphi = \sum_j \hbar^{\frac{1-m_j}{2}} A_{M_j} e^{\frac{i}{\hbar} S_{M_j}} \{1 + O(\hbar)\}, \quad (1.35)$$

where  $A_{M_j}$  is an appropriate amplitude which is independent of  $\hbar$ , and  $S_{M_j}$  denotes the constant value of the action  $S(x, x; E)$  on  $M_j$ . If  $\gamma$  is an isolated periodic orbit, the corresponding manifold  $M_j$  is one dimensional, so that it yields a contribution of  $O(\hbar^0)$  to  $\text{Tr } G_{po}^\varphi$ . A  $k$ -parameter family of periodic orbits leads to  $m_j = k + 1$  and hence contributes  $O(\hbar^{-\frac{k}{2}})$ . In particular, in a classically integrable system each invariant torus is of dimension  $d$  so that its corresponding contribution is  $O(\hbar^{\frac{1-d}{2}})$ .

When the classical dynamics is such that all periodic orbits are isolated, and thus all connected manifolds  $M_j$  of stationary points are one dimensional, the amplitudes  $A_{M_j}$  can be calculated explicitly. The result being due to Gutzwiller can, e.g., be found in [9, 2] and reads

$$\text{Tr } G_{po}^\varphi = \frac{1}{2\pi} \sum_\gamma \frac{T_{\gamma_p} \hat{\varphi}(T_\gamma) e^{\frac{i}{\hbar} S_\gamma - i\frac{\pi}{2} \tilde{\mu}_\gamma}}{|\det(M_\gamma - \mathbb{1})|^{\frac{1}{2}}} \{1 + O(\hbar)\}. \quad (1.36)$$

Here the sum extends over all classical periodic orbits with energy  $E$  such that their periods  $T_\gamma$  are contained in the support of  $\hat{\varphi}(t)$ .  $T_{\gamma_p}$  then denotes the primitive period corresponding to  $\gamma$ , i.e., the period of the primitive periodic orbit  $\gamma_p$  attached to  $\gamma$ . If  $\gamma$  is a  $k$ -fold traversal of  $\gamma_p$  we write  $\gamma = \gamma_p^k$  and obviously find that  $T_\gamma = kT_{\gamma_p}$ . Notice that  $k$  can be both positive and negative, corresponding to traversals of the primitive orbit in both directions. The quantity  $M_\gamma$  appearing in (1.36) denotes the monodromy matrix, or stability matrix, of the periodic orbit  $\gamma$ . It is given as a linearization of the Poincaré recurrence map on a surface of section transversal to  $\gamma$  in phase space. Collecting now the contributions (1.22) and (1.36) to (1.11) yields the regularized Gutzwiller Trace Formula (GTF)

$$\sum_n \varphi\left(\frac{E_n - E}{\hbar}\right) = \frac{\text{vol}(\Omega_E)}{(2\pi\hbar)^{d-1}} \frac{\hat{\varphi}(0)}{2\pi} \{1 + O(\hbar)\}$$



$$+ \sum_{\gamma_p} \sum_{k \neq 0} \frac{\hat{\varphi}(kT_{\gamma_p})}{2\pi} \frac{T_{\gamma_p} e^{\frac{i}{\hbar} k S_{\gamma_p} - i \frac{\pi}{2} k \tilde{\mu}_{\gamma_p}}}{|\det(M_{\gamma_p}^k - \mathbb{1})|^{\frac{1}{2}}} \{1 + O(\hbar)\} , \quad (1.37)$$

where the sum over all periodic orbits with energy  $E$  has been replaced by a sum over primitive orbits and their  $k$ -fold repetitions. Exploiting the relation (1.13) now allows to obtain a periodic-orbit representation for the spectral density [9, 2],

$$d(E) = \frac{\text{vol}(\Omega_E)}{(2\pi\hbar)^d} \{1 + O(\hbar)\} + \frac{1}{\pi\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{T_{\gamma_p} \cos\left(\frac{k}{\hbar} S_{\gamma_p} - \frac{\pi}{2} k \tilde{\mu}_{\gamma_p}\right)}{|\det(M_{\gamma_p}^k - \mathbb{1})|^{\frac{1}{2}}} \{1 + O(\hbar)\} . \quad (1.38)$$

At this place a remark seems to be in order. The test function  $\hat{\varphi}(t)$  cuts off the sum over periodic orbits in (1.37) since it has a compact support, i.e., it vanishes outside a finite interval  $[T_1, T_2]$ . In hyperbolic classical dynamical systems the number of periodic orbits with periods not exceeding  $T$  is finite and obeys the asymptotic law

$$\mathcal{N}(T) := \#\{\gamma; 0 < T_{\gamma} \leq T\} \sim \frac{e^{h_{top}T}}{h_{top}T} , \quad T \rightarrow \infty . \quad (1.39)$$

Here  $h_{top} > 0$  denotes the topological entropy of the classical dynamics on the energy shell  $\Omega_E$ . Therefore the sum on the r.h.s. of (1.37) is actually of finite length. However, the corresponding sum in (1.38) is infinite, and due to the exponential proliferation (1.39) of the number of periodic orbits it indeed is divergent. But this divergence is to be expected because the sum approximates the spectral density which is a singular object. Thus, (1.38) should be understood as a formal relation whose actual meaning is provided by (1.37).

A glance at (1.38) reveals that for each value  $E$  at which the spectral density shall be evaluated one has to determine the classical periodic orbits of energy  $E$ , their actions, stabilities, etc. In general this is a formidable task which renders an application of the trace formula (1.38) for  $d(E)$  almost impossible. However, for a certain class of dynamical systems considerable simplifications emerge in that periodic orbits need only be calculated at a fixed reference energy  $E_0$ . A rather simple scaling relation then determines all required quantities at arbitrary values  $E$  of the energy. Such a mechanism to apply requires a mechanical similarity which allows to associate uniquely a periodic orbit at each value  $E$ , given one at  $E_0$ . Furthermore, the actions of periodic orbits have to be homogeneous functions of the energy,  $S_{\gamma}(\lambda E) = \lambda^{\alpha} S_{\gamma}(E)$  for all  $\lambda > 0$ . One can now fix  $E_0$  and obtain the energy dependence of  $S_{\gamma}(E)$  as

$$S_{\gamma}(E) = E^{\alpha} E_0^{-\alpha} S_{\gamma}(E_0) . \quad (1.40)$$

It then proves useful to discuss the GTF in terms of the scaling variable  $E^{\alpha}$ . Examples for systems with a mechanical similarity and homogeneous actions can be found among all the cases 1.-3. mentioned at the beginning of section 1. Hamilton functions  $H(p, x) = \frac{p^2}{2m} + V(x)$  with scaling potentials,  $V(\lambda x) = \lambda^{\kappa} V(x)$  for all  $\lambda > 0$ , yield homogeneous actions of degree  $\alpha = \frac{1}{2} + \frac{1}{\kappa}$ . Billiards and geodesic flows on Riemannian manifolds always show the property (1.40) with  $\alpha = \frac{1}{2}$ , since  $S_{\gamma}(E) = \int_{\gamma} p dx = \sqrt{2mE} l_{\gamma}$ , where  $l_{\gamma}$  denotes the geometric length of the periodic orbit  $\gamma$ . We remark that in quantum systems whose classical limit is scaling in the above sense, the semiclassical limit  $\hbar \rightarrow 0$  obviously is completely equivalent to the limit  $E^{\alpha} \rightarrow \infty$ . Once  $\alpha > 0$  this in turn is equivalent to the high-energy limit  $E \rightarrow \infty$ . An error term of the form  $O(\hbar^k)$ ,  $\hbar \rightarrow 0$ , can therefore be replaced by  $O(E^{-\alpha k})$ ,  $E^{\alpha} \rightarrow \infty$ .

As an example, let us now discuss the GTF for quantum billiards with two degrees of freedom in some more detail. For simplicity, units will be chosen such that  $2m = 1$ , and by abuse of notation we write  $p := +\sqrt{E} \geq 0$  so that  $S_{\gamma}(E) = p l_{\gamma}$ . The spectral density shall then be expressed in terms of the momentum variable  $p$ ,

$$d(E) = \sum_n \delta(E - E_n) = \sum_n \delta(p^2 - p_n^2) = \frac{1}{2p} \sum_n \delta(p - p_n) =: \frac{1}{2p} \tilde{d}(p) . \quad (1.41)$$

The trace formula for the spectral density  $\tilde{d}(p)$  thus reads

$$\tilde{d}(p) = \tilde{d}_0(p) + \frac{1}{\pi\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{l_{\gamma_p} \cos\left(\frac{p}{\hbar} k l_{\gamma_p} - \frac{\pi}{2} k \tilde{\mu}_{\gamma_p}\right)}{|\det(M_{\gamma_p}^k - \mathbf{1})|^{\frac{1}{2}}} \left\{1 + O\left(\frac{1}{p}\right)\right\}, \quad p \rightarrow \infty, \quad (1.42)$$

where  $\tilde{d}_0(p)$  denotes the analogue to the first term on the r.h.s. of (1.38) and provides a mean behaviour for the spectral density. If  $d = 2$  the leading order asymptotics follows from (1.22). In addition, the subleading term for  $\tilde{d}_0(p)$  is also known,

$$\tilde{d}_0(p) = \frac{A}{2\pi\hbar^2} p \mp \frac{L}{4\pi\hbar} + O\left(\frac{1}{p}\right), \quad p \rightarrow \infty, \quad (1.43)$$

where  $A$  denotes the area of the billiard domain, and  $L$  the length of its boundary. The negative and positive sign correspond to Dirichlet and Neumann boundary conditions for the Laplacian, respectively.

The sum over periodic orbits in (1.42) does not converge, as has already been discussed for the corresponding sum in (1.38). Again, (1.42) shall rather be viewed as a distributional relation that has to be evaluated on suitable test functions in order to yield an analogue of (1.37). Due to the scaling property on does, however, no longer need an external parameter fixing the energy at which the periodic orbit sum is to be evaluated. Let us therefore choose a smooth test function  $h(p)$ , whose further properties will follow from a subsequent discussion. Since the variable  $p$  derives from the energy variable  $E = p^2$ ,  $h(p)$  should be chosen as an even function,  $h(p) = h(-p)$ , so that it can also be defined for negative  $p$ . An evaluation of (1.43) on such a test function thus leads to a regularized trace formula for billiards,

$$\sum_n h(p_n) = \int_0^\infty \tilde{d}_0(p) h(p) dp + \frac{1}{\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{l_{\gamma_p} \mathcal{F}_{\gamma_p^k}[h]\left(\frac{kl_{\gamma_p}}{\hbar}\right)}{|\det(M_{\gamma_p}^k - \mathbf{1})|^{\frac{1}{2}}} \{1 + O(\hbar)\}, \quad \hbar \rightarrow 0, \quad (1.44)$$

where the following definition enters,

$$\mathcal{F}_\gamma[h](u) := \frac{1}{\pi} \int_0^\infty h(p) \cos\left(pu - \frac{\pi}{2} \tilde{\mu}_\gamma\right) dp. \quad (1.45)$$

Since

$$\cos\left(pu - \frac{\pi}{2} \tilde{\mu}_\gamma\right) = \begin{cases} (-1)^{\frac{\tilde{\mu}_\gamma}{2}} \cos(pu) & , \quad \tilde{\mu}_\gamma \text{ even} , \\ (-1)^{\frac{\tilde{\mu}_\gamma-1}{2}} \sin(pu) & , \quad \tilde{\mu}_\gamma \text{ odd} , \end{cases} \quad (1.46)$$

in case  $\tilde{\mu}_{\gamma_p}$  is even this yields

$$\mathcal{F}_{\gamma_p^k}[h](u) = e^{-i\frac{\pi}{2}k\tilde{\mu}_{\gamma_p}} \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(p) e^{ipu} dp =: e^{-i\frac{\pi}{2}k\tilde{\mu}_{\gamma_p}} g(u). \quad (1.47)$$

The version of the GTF that emerges in this situation hence reads

$$\sum_n h(p_n) = \int_0^\infty \tilde{d}_0(p) h(p) dp + \frac{1}{\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{l_{\gamma_p} e^{-i\frac{\pi}{2}k\tilde{\mu}_{\gamma_p}} g\left(\frac{kl_{\gamma_p}}{\hbar}\right)}{|\det(M_{\gamma_p}^k - \mathbf{1})|^{\frac{1}{2}}} \{1 + O(\hbar)\}, \quad \hbar \rightarrow 0, \quad (1.48)$$

and was in this form given in [10].

The criteria that fix the class of test functions  $h(p)$  to be admitted in the trace formula (1.48) derive from the necessity that all terms entering (1.48) be finite. First of all, the leading asymptotic behaviour (1.43) of the spectral density yields Weyl's law

$$N(p) := \#\{n; 0 \leq p_n \leq p\} \sim \frac{A}{4\pi\hbar^2} p^2, \quad p \rightarrow \infty, \quad (1.49)$$

upon integrating  $\tilde{d}(p)$  once. Since obviously  $N(p_n) = n$ , (1.49) can be inverted to observe  $p_n \sim \text{const.} \sqrt{n}$ ,  $n \rightarrow \infty$ . This requires the test function to obey  $h(p) = O(|p|^{-2-\delta})$ ,  $|p| \rightarrow \infty$ , for some  $\delta > 0$ , in order to render the sum on the l.h.s. of (1.48) convergent. Due to (1.43) the same condition also ensures the convergence of the first term on the r.h.s. of (1.48). The exponential proliferation (1.39) of the number of periodic orbits leads one to anticipate that the Fourier transform  $g(u)$  of the test function  $h(p)$  has to be required to decrease exponentially for  $|u| \rightarrow \infty$ , i.e.,  $g(u) = O(e^{-(\sigma+\varepsilon)|u|})$  for some  $\varepsilon > 0$  and some characteristic constant  $\sigma > 0$  that is determined by the distribution of classical periodic orbits. Due to the definition (1.47) of  $g(u)$  as a Fourier transform the exponential asymptotic estimate for  $g(u)$  is clearly equivalent to demand that the test function  $h(p)$  itself be holomorphic in a strip  $|\text{Im } p| \leq \sigma + \varepsilon$ .

A means to characterize the constant  $\sigma$  that determines the strip of holomorphy to be demanded for the test function  $h(p)$  is provided by the thermodynamic formalism, see [11, 12]. In the latter theory one introduces the Ruelle zeta function

$$\zeta_\beta(s) := \prod_{\gamma_p} \left(1 - e^{-sl_{\gamma_p} - \beta u_{\gamma_p}}\right)^{-1}, \quad (1.50)$$

where  $e^{u_{\gamma_p}} > 1$  denotes the modulus of one of the eigenvalues of the monodromy matrix  $M_{\gamma_p}$ ; the other eigenvalue of this matrix then is of modulus  $e^{-u_{\gamma_p}} < 1$ . Thus the product over primitive periodic orbits in (1.50) converges for those complex  $s$  such that  $\text{Re } s$  is large enough; here  $\beta$  is considered as a parameter and is kept fixed. Indeed, one denotes the abscissa of convergence for (1.50) as  $P(\beta)$ , so that the condition  $\text{Re } s > P(\beta)$  defines the right half-plane where the Euler product in (1.50) converges absolutely. Thus,  $P(\beta)$  has the following representation,

$$P(\beta) = \inf \left\{ t \in \mathbb{R}; \sum_{\gamma_p} e^{-tl_{\gamma_p} - \beta u_{\gamma_p}} < \infty \right\}. \quad (1.51)$$

In the thermodynamic formalism  $P(\beta)$  is known as the topological pressure of the classical dynamics. Once the latter is hyperbolic,  $P(\beta)$  is shown to be of a certain universal form; this in particular implies that  $P(\frac{1}{2}) > 0$ . Now, since  $|\det(M_{\gamma_p} - \mathbf{1})|^{-\frac{1}{2}} = e^{-\frac{1}{2}u_{\gamma_p}} [1 + o(1)]$ , the condition which ensures an absolutely convergent sum over periodic orbits in (1.48) reads

$$\sum_{\gamma_p} l_{\gamma_p} e^{-\frac{1}{2}u_{\gamma_p}} \left| g\left(\frac{l_{\gamma_p}}{\hbar}\right) \right| < \infty. \quad (1.52)$$

Hence, (1.52) is satisfied as long as  $g(u) = O(e^{-(P(\frac{1}{2})+\varepsilon)\hbar|u|})$  for  $|u| \rightarrow \infty$ , and thus one identifies  $\sigma = \hbar P(\frac{1}{2}) > 0$ .

In summary one concludes that the trace formula for billiards as given in (1.48) contains only finite quantities, once the test function  $h(p)$  satisfies the three conditions

1.  $h(p) = h(-p)$ ,
2.  $h(p)$  is holomorphic in the strip  $|\text{Im } p| \leq \sigma + \varepsilon$  for some  $\varepsilon > 0$ , where  $\sigma$  is a positive characteristic constant,  $\sigma = \hbar P(\frac{1}{2})$ ,
3.  $h(p) = O(|p|^{-2-\delta})$  for  $|p| \rightarrow \infty$ , where  $\delta > 0$  is arbitrary.

At this point we would like to add a remark on the mathematical status of the trace formula. Gutzwiller originally derived [8, 9] his trace formula for the spectral density  $d(E)$ , and obtained the relation (1.38). Slightly later, but seemingly completely independently, similar investigations were performed in the mathematical community. The first rigorous proof of a trace formula was given in a special case by Duistermaat and Guillemin [13]: The ‘quantum Hamiltonian’ they considered was

an elliptic positive pseudodifferential operator of degree one on a compact smooth manifold without boundary; an example for such an operator is the square root of minus the Laplacian on the manifold. The corresponding classical dynamics then is generated by a ‘Hamiltonian function’ which is given by the square root of the kinetic energy for a single particle. Since thus the mechanical similarity as discussed above applies, one can choose  $E_0 = 1$  as a reference energy. Then the classical dynamics generated by the kinetic energy term and its square root, respectively, coincide on  $\Omega_{E_0=1}$ . Translating now the result of [13] into a relation for the spectral density  $\tilde{d}(p)$ , one recovers (1.42) with  $\hbar = 1$ . Subsequently, generalizations in several directions were achieved. Finally, under certain assumptions on the class of quantum Hamiltonians and on the regularity of the classical dynamics, the semiclassical trace formula as given in (1.37) was proven [14, 15]. In principle, the strategy employed in [13, 14, 15] was to start with an ansatz like (1.14) and then to give the integral on the r.h.s. as well as the expansion (1.16) a precise mathematical meaning. Thereafter the multiple integral in (1.19) could be evaluated essentially by employing the method of stationary phase. In our presentation, we tried to discuss the trace formula in a manner that somehow interpolates between Gutzwiller’s original investigation and the rigorous mathematical treatment, which itself requires an extensive technical apparatus.

## 1.2 Some Applications of Trace Formulae

In order to illustrate the use of semiclassical trace formulae in quantum chaos, we are now going to discuss some of their applications. For simplicity the following considerations will be restricted to the cases 2.-3. of the list at the beginning of section 1. Hence the relevant form of the trace formula is given by (1.48). To begin with let us study the test function  $h(p) := e^{-p^2 t}$ ,  $t > 0$ , which yields on the l.h.s. of the trace formula the trace of the heat kernel,

$$\Theta_{\hat{H}}(t) = \text{Tr } e^{-\hat{H}t} = \sum_n e^{-p_n^2 t} . \quad (1.53)$$

For quantum billiards as well as for Laplacians on Riemannian manifolds the asymptotic behaviour of  $\Theta_{\hat{H}}(t)$  as  $t \rightarrow 0$ , or equivalently as  $\hbar \rightarrow 0$ , is well known to yield [16]

$$\Theta_{\hat{H}}(t) = \frac{A}{4\pi\hbar^2 t} \mp \frac{L}{8\sqrt{\pi t\hbar}} + c_0 + \sum_{n=1}^{N-1} c_n \hbar^n t^{\frac{n}{2}} + O\left(\hbar^N t^{\frac{N}{2}}\right) , \quad (1.54)$$

with some appropriate coefficients  $c_n$  that can in principle be successively determined. The r.h.s. of the trace formula can be evaluated once the Fourier transform  $g(u) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{u^2}{4t}}$  of the test function is inserted in (1.48). Thus

$$\Theta_{\hat{H}}(t) = \int_0^\infty \tilde{d}_0(p) e^{-p^2 t} dp + \frac{1}{\sqrt{4\pi\hbar^2 t}} \sum_{\gamma_p} \sum_{k=1}^\infty \frac{l_{\gamma_p} e^{-i\frac{\pi}{2} k \tilde{\mu}_{\gamma_p}} e^{-\frac{k^2 l_{\gamma_p}^2}{4\hbar^2 t}}}{|\det(M_{\gamma_p}^k - \mathbf{1})|^{\frac{1}{2}}} \{1 + O(\hbar)\} . \quad (1.55)$$

For  $\hbar \rightarrow 0$  each term in the sum over periodic orbits is of  $O(\hbar^\infty)$  so that in (1.54) all power-like contributions in  $\hbar$  have to derive from  $\tilde{d}_0(p)$ . A term-by-term inversion then yields the semiclassical expansion

$$\tilde{d}_0(p) = \frac{Ap}{2\pi\hbar^2} \mp \frac{L}{4\pi\hbar} + O(\hbar) . \quad (1.56)$$

As remarked earlier, the asymptotics for  $\hbar \rightarrow 0$  can be converted into an asymptotics for  $p \rightarrow \infty$ , so that (1.56) exactly reproduces (1.43). In fact, the above reasoning is a proper justification for (1.43).

Among the first applications of trace formulae in quantum chaos one finds efforts to introduce semiclassical quantization rules for classically chaotic systems. These should serve as substitutes for the semiclassical EBK-quantization scheme which only applies to classically integrable systems. A

first guess of a quantization rule could be to use the representation (1.42) for the spectral density  $\tilde{d}(p)$ : one has to find classical periodic orbits and to calculate their lengths, stabilities, and Maslov phases. Upon evaluating the r.h.s. of (1.42) with these data one would then approximate  $\tilde{d}(p)$  and thus be able to identify the singularities of the spectral density. However, due to the lack of convergence of the periodic orbit sum in (1.42) one has no good control over the quality of the approximation that occurs by cutting off the sum after finitely many terms. Therefore a regularization is called for which renders the periodic orbit sum absolutely convergent. One hence is advised to consult the smoothed trace formula (1.48) with a suitable test function. Having in mind to approximate the spectral density, as a natural choice for an admissible test function a Gaussian

$$h(p') = \frac{1}{\varepsilon\sqrt{\pi}} \left[ e^{-\frac{(p'-p)^2}{\varepsilon^2}} + e^{-\frac{(p'+p)^2}{\varepsilon^2}} \right] \quad (1.57)$$

was suggested in [17], so that the l.h.s. of (1.48) yields a smoothed spectral density. This approaches  $\tilde{d}(p)$  in the limit  $\varepsilon \rightarrow 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \sum_n h(p_n) = \sum_n [\delta(p_n - p) + \delta(p_n + p)] = \tilde{d}(p) , \quad (1.58)$$

if  $p > 0$ . The second Gaussian in (1.57) is necessary to yield an even test function,  $h(-p') = h(p')$ . If  $p$  is not too small, the first Gaussian yields peaks of height  $\frac{1}{\varepsilon\sqrt{\pi}}$  at each  $p = p_n$ , and the second Gaussian only adds a negligible value. Thus, upon scanning  $p$  one can detect the quantum energies as  $E_n = p_n^2$ . Now the r.h.s. of (1.48) can be evaluated with the Fourier transform  $g(u) = \frac{1}{\pi} \cos(pu) e^{-\frac{\varepsilon^2 u^2}{4}}$ . Therefore the periodic orbit sum

$$\frac{1}{\pi\hbar} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{l_{\gamma_p} e^{-i\frac{\pi}{2}k\tilde{\mu}_{\gamma_p}} \cos\left(\frac{p}{\hbar}kl_{\gamma_p}\right)}{|\det(M_{\gamma_p}^k - \mathbf{1})|^{\frac{1}{2}}} e^{-\frac{\varepsilon^2}{4}k^2l_{\gamma_p}^2} \quad (1.59)$$

clearly converges absolutely due to the Gaussian suppression of the exponential proliferation (1.39) of the number of terms entering (1.59). Moreover, one can explicitly observe that the limit  $\varepsilon \rightarrow 0$  applied to (1.59) recovers the periodic orbit sum in (1.42).

Integrating the spectral density once with respect to  $p$  yields the spectral staircase (1.49). The trace formula with the test function (1.57) will therefore now be integrated in  $p$  over the interval  $[0, q]$ . On the l.h.s. one then obtains the approximation

$$N_{\varepsilon}(q) = \sum_{0 \leq p_n < q} \left[ 1 + O\left(\varepsilon e^{-\frac{c}{\varepsilon^2}}\right) \right] + \sum_{p_n = q} \left[ \frac{1}{2} + O\left(\varepsilon e^{-\frac{c}{\varepsilon^2}}\right) \right] + \sum_{p_n > q} O\left(\varepsilon e^{-\frac{c}{\varepsilon^2}}\right) \quad (1.60)$$

to the spectral staircase. However, in the limit  $\varepsilon \rightarrow 0$  the expression (1.60) approaches the symmetrized staircase function

$$N_0(q) = \lim_{\delta \rightarrow 0} \frac{1}{2} [N(q + \delta) + N(q - \delta)] , \quad (1.61)$$

which is identical to  $N(q)$  whenever  $q \neq p_n$ . Otherwise one obviously observes that  $N(p_n) = n - \frac{1}{2}$ . This allows to set up the quantization rule

$$\cos[\pi N_0(q)] = 0 \quad \Leftrightarrow \quad q \in \{p_1, p_2, p_3, \dots\} . \quad (1.62)$$

According to (1.59) the leading periodic orbit contribution to (1.60) reads

$$N_{\varepsilon,fl}(q) = \frac{1}{\pi} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{e^{-i\frac{\pi}{2}k\tilde{\mu}_{\gamma_p}}}{k} \frac{\sin\left(\frac{q}{\hbar}kl_{\gamma_p}\right)}{|\det(M_{\gamma_p}^k - \mathbf{1})|^{\frac{1}{2}}} e^{-\frac{\varepsilon^2}{4}k^2l_{\gamma_p}^2} \{1 + O(\hbar)\} , \quad (1.63)$$

which again is an absolutely convergent sum as long as  $\varepsilon > 0$ . In [18] it was suggested to use (1.63) in (1.62) to obtain an approximate semiclassical quantization rule. It was furthermore demonstrated in several examples that this procedure yields good numerical approximations to the quantum energies  $E_n = p_n^2$ , and that this scheme is quantitatively superior to the direct use of a Gaussian test function (1.57) in the trace formula.

Inspecting the periodic orbit sum (1.63) one observes that, viewed as a function of  $q$ , it constitutes a superposition of sine-oscillations with wave lengths  $\frac{2\pi\hbar}{kl_{\gamma p}}$ . Adding those terms finally yields  $N_\varepsilon(q)$ . One can now ask how many of these terms become efficient when one is interested in resolving all quantum energies  $E_n \leq p^2$  with the quantization rule (1.62), see [19] for related discussions. At the point  $p$  the mean density of zeros of the expression  $\cos[\pi N_0(p)]$  is given by (1.43) so that their mean separation  $\Delta p$  asymptotically reads  $\Delta p \sim \frac{2\pi\hbar^2}{Ap}$ . In order to effectively resolve the zeros  $p_n$  in the vicinity of  $p$  with the help of the periodic orbit sum (1.63) one hence has to add all oscillatory terms with wave lengths down to  $\Delta p$ , i.e., one needs all periodic orbits  $\gamma = \gamma_p^k$  with  $\frac{2\pi\hbar}{kl_{\gamma p}} \geq \Delta p \sim \frac{2\pi\hbar^2}{Ap}$ . Thus, a cut-off at approximately  $kl_{\gamma p} \approx \frac{Ap}{\hbar}$  ensures that one has taken all relevant contributions into account. Since the exponential proliferation (1.39) of the number of periodic orbits reads in terms of their geometrical lengths

$$\tilde{N}(l) = \# \{ \gamma; l_\gamma \leq l \} \sim \frac{e^{\tau l}}{\tau l}, \quad l \rightarrow \infty, \quad (1.64)$$

where  $\tau$  denotes a scaled topological entropy which is independent of  $p$ , the number  $N_p$  of terms to be included in (1.63) in order to resolve all quantum energies with  $E_n \leq p^2$  is given by

$$N_p \sim \frac{\hbar}{A\tau p} e^{\frac{A\tau}{\hbar} p}, \quad p \rightarrow \infty. \quad (1.65)$$

Recalling now the asymptotics (1.49), the number  $N_n$  of periodic orbits required to resolve the lowest  $n$  quantum energies thus derives from (1.65) to behave as

$$N_n \sim \frac{1}{2\tau\sqrt{\pi An}} e^{2\tau\sqrt{\pi An}}, \quad n \rightarrow \infty. \quad (1.66)$$

Therefore, the computational effort to obtain more and more quantum energies by using the semiclassical quantization rule grows enormously.

Similar considerations apply to all quantization procedures that are based on semiclassical trace formulae since the structure of the periodic orbit sums involved together with the exponential proliferation of the number of periodic orbits will always produce estimates of the effort to be spent which are similar to (1.66); one will certainly not achieve an improvement that comes down to a power-law behaviour for  $N_n$ . However, in some specific systems, especially in classically chaotic scattering systems, it might happen that one can resolve a certain number of zeros using only a few periodic orbits. In these situations the specific features of the systems cause the number of required periodic orbits to blow up only at considerably large  $p_n$ . In this context one should bear in mind that (1.66) is only an asymptotic statement. The principal effect, resulting in the estimate (1.66), is however inherent in semiclassical quantization procedures based on trace formulae and cannot be simply overcome. At this point also the analogy between the zeros of the Riemann zeta function, see below, and quantum energies of classically chaotic systems breaks down since the density of Riemann zeros is considerably lower than (1.43). This results in a power-law estimate for  $N_n$ , and that has to be compared with (1.66), see [19] for a brief account. Basically the same problem occurred in the mathematical literature on the Selberg trace formula, where it was noted that well known estimates that hold in the theory of the Riemann zeta function could not be carried over to the Selberg zeta function, see [20] for an extensive discussion.

As a consequence, various studies have revealed that it is possible to calculate a certain number, say some ten, quantum energies from classical data employing semiclassical trace formulae. However, the

huge numerical effort to be spent in order to increase the output of the number of eigenvalues prohibits to apply these semiclassical methods on a large scale. It turns out that numerical methods to solve the eigenvalue problem of the quantum Hamiltonian directly are far more efficient than semiclassical ones, if one is interested in obtaining a large number of quantum energies.

### 1.3 The Selberg Trace Formula and the Selberg Zeta Function

The trace formulae discussed so far essentially yield the leading semiclassical approximation to the spectral density. In addition, one has to cope with a further approximation in applications because in practise one can only take finitely many terms of a periodic orbit sum into account. Thus, it is often difficult to trace back numerical inaccuracies to either one of the approximations applied. Luckily there exist cases in which an exact trace formula is at hand, which therefore does not include any semiclassical approximation. The classical systems to be considered are geodesic motions on manifolds of constant negative curvature, and their quantizations are provided by the quantum Hamiltonians  $\hat{H} = -\frac{\hbar^2}{2m}\Delta$ , where  $\Delta$  denotes the Laplace-Beltrami operator for the manifold. Due to the negative curvature the classical dynamics show a strong chaotic behaviour, and thus these models serve as convenient playgrounds for quantum chaos.

In the simplest case of  $d = 2$  degrees of freedom one has to deal with the geometry of hyperbolic surfaces, see for example [21, 22]. Here we choose as a model for two dimensional hyperbolic geometry the unit disc

$$\mathcal{D} := \{z = x + iy \in \mathbb{C}; |z| < 1\} \quad (1.67)$$

endowed with the Poincaré metric  $ds^2 = 4(1 - x^2 - y^2)^{-1}(dx^2 + dy^2)$  of constant negative Gaussian curvature  $K = -1$ . The classical dynamics of a single particle on  $\mathcal{D}$  is that of a free motion along geodesics. The latter are those circular arcs and straight lines in  $\mathcal{D}$  that are perpendicular to the unit circle. The hyperbolic distance  $d(z, w)$  between two points  $z \neq w$  in  $\mathcal{D}$  is measured with the Poincaré metric  $ds^2$  and is given as the hyperbolic length of the unique geodesic arc connecting  $z$  and  $w$ . In explicit terms it reads

$$\cosh d(z, w) = 1 + \frac{2|z - w|^2}{(1 - |z|^2)(1 - |w|^2)} . \quad (1.68)$$

From this relation one immediately concludes that any point  $w$  on the unit circle is infinitely far away from any point  $z$  in the interior of  $\mathcal{D}$ .

Due to the latter observation the Poincaré unit disc itself is an infinitely extended two dimensional hyperbolic space. In order to yield compact and closed surfaces, and thus examples for the case 3. of the list at the beginning of section 1, one has to confine oneself to bounded domains in the unit disc supplied with appropriate boundary identifications. In as much as a flat torus can be constructed from a parallelogram in the euclidean plane with opposite edges identified, one can obtain compact surfaces with a metric of constant negative curvature by an analogous construction. The boundary identifications that produce a torus out of a parallelogram are translations on the euclidean plane. The group generated by the two identifications of a torus then is a discrete subgroup of the group of motions of the plane, i.e., the group of translations and rotations. The analogous transformations on the Poincaré disc are the Möbius transformations

$$z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} , \quad (1.69)$$

with  $|\alpha|^2 - |\beta|^2 = 1$ , which indeed map the interior of  $\mathcal{D}$  to itself. Introducing the matrix group  $SU(1, 1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}; |\alpha|^2 - |\beta|^2 = 1 \right\}$ , a composition of two Möbius transformations (1.69) corresponds to a matrix multiplication in  $SU(1, 1)$ . The latter group is therefore the analogue of the group of motions on the euclidean plane. Given now a suitable bounded domain  $\mathcal{F} \subset \mathcal{D}$ , with finitely many geodesic arcs as its boundary components, the Möbius transformations (1.69) identifying pairs of edges generate a discrete subgroup  $\Gamma \subset SU(1, 1)$ .

The quantum Hamiltonian  $\hat{H} = -\frac{\hbar^2}{2m}\Delta$  requires to know the Laplace-Beltrami operator  $\Delta$  for the Poincaré disc  $\mathcal{D}$ ,

$$-\Delta = -\frac{1}{4} (1 - x^2 - y^2) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \quad (1.70)$$

If this operator is defined on the smooth functions on  $\mathcal{D}$  it has a continuous spectrum  $[\frac{1}{4}, \infty)$ . Due to the peculiar value of the bottom of the spectrum one parametrizes the quantum energies as  $E = \frac{\hbar^2}{2m}(p^2 + \frac{1}{4})$ , with  $p \geq 0$ . For the following we choose units such that  $\hbar = 1 = 2m$ , and hence  $E = p^2 + \frac{1}{4}$ . If one now constructs a compact and closed surface from a suitable domain  $\mathcal{F} \subset \mathcal{D}$  as described above, one imposes periodic boundary conditions on the eigenfunctions of  $\hat{H} = -\Delta$ . Let  $g_1, \dots, g_k$  be the Möbius transformations (1.69) identifying pairs of edges of  $\mathcal{F}$ , then one demands that  $\psi(g_j z) = \psi(z)$ ,  $j = 1, \dots, k$ . Since the transformations  $g_1, \dots, g_k$  generate the discrete group  $\Gamma \subset SU(1, 1)$ , the periodicity extends to all elements of  $\Gamma$ ,  $\psi(gz) = \psi(z)$  for all  $g \in \Gamma$ . The Laplacian now being defined on a compact surface has a discrete spectrum  $0 = E_0 < E_1 \leq E_2 \leq \dots$ . If one defines  $p_n := \sqrt{E_n - \frac{1}{4}}$ , the spectral staircase satisfies the Weyl asymptotics (1.49).

In the 1950's Selberg [23] employed the above constructions to obtain the Selberg Trace Formula (STF)

$$\sum_n h(p_n) = \frac{A}{2\pi} \int_0^\infty p \tanh(\pi p) h(p) dp + \sum_{\gamma_p} \sum_{k=1}^\infty \frac{l_{\gamma_p} g(kl_{\gamma_p})}{2 \sinh(\frac{kl_{\gamma_p}}{2})}, \quad (1.71)$$

see also [20]. Here  $A$  denotes the area of the given surface, measured with the Poincaré metric, and the sum over the  $\gamma_p$ 's extends over all primitive periodic orbits of the associated classical dynamics. Since the latter are given by the geodesic motion, the periodic orbits are the closed geodesics on the compact surface. The test function  $h(p)$  is required to fulfill

1.  $h(-p) = h(p)$ ,
2.  $h(p)$  is holomorphic in the strip  $|\operatorname{Im} p| \leq \frac{1}{2} + \varepsilon$  for some  $\varepsilon > 0$ ,
3.  $h(p) = O(|p|^{-2-\delta})$  for  $|p| \rightarrow \infty$ , where  $\delta > 0$  is arbitrary.

Furthermore,  $g(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(p) e^{ipu} dp$  is the Fourier transform of the test function  $h(p)$ .

A comparison of the STF (1.71) with the trace formula (1.48) reveals that both relations are almost identical. If one were to set up the trace formula (1.48) for the systems presently under study one would first of all notice that due to the STF all higher-order corrections to the periodic orbit sum which are not explicitly contained in (1.48) indeed vanish. One is then furthermore able to read off from the STF a number of quantities entering the semiclassical trace formula:

1. The mean spectral density is completely known,

$$\tilde{d}_0(p) = \frac{A}{2\pi} p \tanh(\pi p) \sim \frac{A}{2\pi} p + O(e^{-2\pi p}), \quad p \rightarrow \infty.$$

Since the surface has no boundary, in a comparison with (1.43) one has to choose  $L = 0$ .

2. The eigenvalues of the monodromy matrices  $M_\gamma$  are given by  $e^{\pm u_\gamma} = e^{\pm l_\gamma}$ , that is  $u_\gamma = l_\gamma$ , for all periodic orbits  $\gamma$ .
3. No Maslov phases occur,  $e^{-i\frac{\pi}{2}\tilde{\mu}_\gamma} = 1$ .
4. The width of the strip of holomorphy demanded for  $h(p)$  is explicitly known, and hence  $P(\frac{1}{2}) = \frac{1}{2}$ . Indeed, a further study of the classical dynamics yields that the topological pressure is linear,  $P(\beta) = 1 - \beta$ .



Of course, all of the applications discussed for the semiclassical trace formula in section 1.2 carry over to the STF.

All the information on the distribution of classical periodic orbits and on the quantum energies which is contained in the STF can be encoded in a function of the complex variable  $s = \frac{1}{2} - ip$ , where  $p = \sqrt{E - \frac{1}{4}}$  is the momentum variable introduced above, but now extended to the whole complex plane, see [23, 20]. This Selberg zeta function

$$Z(s) := \prod_{\gamma_p} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)l_{\gamma_p}}\right) \quad (1.72)$$

is defined by a product over the classical primitive periodic orbits  $\gamma_p$  which converges in the right half-plane  $\text{Re } s > 1$ . Since the latter condition is equivalent to  $\text{Im } p > \frac{1}{2}$  the domain of convergence excludes the real momentum axis where all quantum energies, with the exception of possibly finitely many, are to be found as  $E_n = p_n^2 + \frac{1}{4}$ .

In order to obtain the analytic properties of the Selberg zeta function one chooses the test function

$$h(p) = \frac{1}{p^2 + (s - \frac{1}{2})^2} - \frac{1}{p^2 + (\sigma - \frac{1}{2})^2} , \quad (1.73)$$

with  $\text{Re } s, \text{Re } \sigma > 1$ , and inserts this into the STF. The Fourier transform of  $h(p)$  reads  $g(u) = \frac{1}{2s-1} e^{-|u|(s-\frac{1}{2})} - \frac{1}{2\sigma-1} e^{-|u|(\sigma-\frac{1}{2})}$  so that the sum over periodic orbits on the r.h.s. of (1.71) can be evaluated,

$$\begin{aligned} \frac{1}{2s-1} \sum_{\gamma_p} \sum_{k=1}^{\infty} \frac{l_{\gamma_p}}{2 \sinh(\frac{kl_{\gamma_p}}{2})} e^{-kl_{\gamma_p}(s-\frac{1}{2})} &= \frac{1}{2s-1} \sum_{\gamma_p} \sum_{n=0}^{\infty} l_{\gamma_p} \sum_{k=1}^{\infty} e^{-(s+n)kl_{\gamma_p}} \\ &= \frac{1}{2s-1} \sum_{\gamma_p} \sum_{n=0}^{\infty} \frac{e^{-(s+n)l_{\gamma_p}}}{1 - e^{-(s+n)l_{\gamma_p}}} \\ &= \frac{1}{2s-1} \frac{d}{ds} \log Z(s) . \end{aligned} \quad (1.74)$$

Extracting the contribution of the lowest eigenvalue  $E_0 = 0$  on the l.h.s. of the STF then yields

$$\begin{aligned} \sum_{n=1}^{\infty} \left[ \frac{1}{E_n + s(s-1)} - \frac{1}{E_n + \sigma(\sigma-1)} \right] &= \frac{1}{\sigma(\sigma-1)} - \frac{1}{s(s-1)} - \frac{A}{2\pi} [\psi(s) - \psi(\sigma)] \\ &\quad + \frac{1}{2s-1} \frac{Z'(s)}{Z(s)} - \frac{1}{2\sigma-1} \frac{Z'(\sigma)}{Z(\sigma)} , \end{aligned} \quad (1.75)$$

where  $\psi(z) = \frac{d}{dz} \log \Gamma(z)$ . Up to now we have restricted the above expressions to the domain  $\text{Re } s, \text{Re } \sigma > 1$  in order to keep the integrals and sums convergent. The relation (1.75), however, allows to analytically continue  $Z(s)$  to all  $s \in \mathbb{C}$ . To this end we perform the limit  $\sigma \rightarrow 1$  and arrive at

$$\sum_{n=1}^{\infty} \left[ \frac{1}{E_n + s(s-1)} - \frac{1}{E_n} \right] = -\gamma_{\Delta} - \frac{1}{s(s-1)} - \frac{A}{2\pi} \psi(s) + \frac{1}{2s-1} \frac{Z'(s)}{Z(s)} , \quad (1.76)$$

where  $\gamma_{\Delta}$  denotes an appropriate constant, see [24]. An integration of this expression with respect to  $s$  results in the product representation [24]

$$Z(s) = Z'(1) s(s-1) e^{s(s-1)\gamma_{\Delta}} \left[ (2\pi)^{(1-s)} e^{-s(s-1)} G(s) G(s+1) \right]^{\frac{A}{2\pi}} \prod_{n=1}^{\infty} \left[ \left( 1 + \frac{s(s-1)}{E_n} \right) e^{-\frac{s(s-1)}{E_n}} \right] \quad (1.77)$$

for the Selberg zeta function, where Barnes' double  $\Gamma$ -function

$$G(z+1) = (2\pi)^{\frac{z}{2}} e^{-\frac{z}{2} - \frac{1+\gamma}{2} z^2} \prod_{n=1}^{\infty} \left[ \left(1 + \frac{z}{n}\right)^n e^{-z + \frac{z^2}{2n}} \right] \quad (1.78)$$

appears. The analytic continuation of the Selberg zeta function based on (1.75) reveals that  $Z(s)$  is an entire holomorphic function with zeros at

- $s = \frac{1}{2} \pm ip_n$  of multiplicity  $d_n$ , when  $E_n = p_n^2 + \frac{1}{4}$ , is an eigenvalue of  $-\Delta$  of multiplicity  $d_n$ ,
- $s = 1$  of multiplicity one,
- $s = 0$  of multiplicity  $\frac{A}{2\pi} + 1$ ,
- $s = -k$ ,  $k \in \mathbb{N}$ , of multiplicity  $\frac{A}{2\pi}(k+1)$ .

We remark that by topological reasons the area of a compact surface of constant negative curvature is such that  $\frac{A}{4\pi}$  is an integer. The analytic structure of  $Z(s)$  can also be directly read off from the product representation (1.77).

If one substitutes  $s \mapsto 1-s$  in (1.77) and subtracts the resulting equation from (1.77) one obtains the functional equation

$$Z(1-s) = Z(s) \exp \left\{ -A \int_0^{s-\frac{1}{2}} u \tan(\pi u) du \right\} \quad (1.79)$$

upon integrating the difference with respect to  $s$ . We now introduce a mean spectral staircase

$$\overline{N}(p) := \int_0^p \tilde{d}_0(q) dq = \frac{A}{2\pi} \int_0^p q \tanh(\pi q) dq, \quad (1.80)$$

and then evaluate the functional equation (1.79) for  $s = \frac{1}{2} - ip$ ,  $p \in \mathbb{R}$ ,

$$Z\left(\frac{1}{2} + ip\right) = Z\left(\frac{1}{2} - ip\right) \exp \left\{ -2\pi i \overline{N}(p) \right\}. \quad (1.81)$$

Thus the function

$$\xi(p) := Z\left(\frac{1}{2} - ip\right) \exp \left\{ -i\pi \overline{N}(p) \right\} = \left| Z\left(\frac{1}{2} - ip\right) \right| \exp \left\{ -i\pi \left[ \overline{N}(p) - \frac{1}{\pi} \arg Z\left(\frac{1}{2} - ip\right) \right] \right\} \quad (1.82)$$

is real when  $p \in \mathbb{R}$  and satisfies  $\xi(-p) = \xi(p)$ . The zeros of  $Z(s)$  at  $s_n = \frac{1}{2} \pm ip_n$  appear as zeros of  $\xi(p)$  at  $\pm p_n$ . Thus, upon increasing  $p$  starting at  $p = 0$ , one successively passes through all  $p_n$ 's related to the quantum energies  $E_n = p_n^2 + \frac{1}{4}$ . Hence, the spectral staircase  $N(p)$  counts all zeros of the function  $\xi(p')$  in the interval  $0 \leq p' \leq p$ . When passing through  $p_n$  the sign of  $\xi(p)$  changes according to the multiplicity  $d_n$  of the eigenvalue  $E_n$ , i.e.,  $\xi(p)$  is multiplied by  $(-1)^{d_n}$ . A comparison with the r.h.s. of (1.82) therefore shows that

$$N(p) = \overline{N}(p) + \frac{1}{\pi} \arg Z\left(\frac{1}{2} + ip\right), \quad (1.83)$$

see also [20] for a detailed discussion of this relation. The r.h.s. of (1.83) can be interpreted as a decomposition of the spectral staircase into a mean part and a part  $N_{fl}(p) = \frac{1}{\pi} \arg Z(\frac{1}{2} + ip)$  describing the fluctuations of the staircase.

At this stage we remark that since the zeros  $p_n$  of the function  $\xi(p)$  are related to the quantum energies, one can introduce the quantization condition

$$\xi(p) \stackrel{!}{=} 0 \quad \Leftrightarrow \quad p^2 + \frac{1}{4} \in \{E_0, E_1, E_2, \dots\}. \quad (1.84)$$

In [25, 26] methods to evaluate for real  $p$  the zeta function, or rather its semiclassical analogue for billiards, which use classical periodic orbits are devised, and then zeros of  $\text{Re } \xi(p)$  are calculated. This procedure leads to good approximations to quantum energies and constitutes an alternative to the quantization rules presented in section 1.2.

In view of the previous discussion one notices that the Selberg zeta function  $Z(s)$  enjoys many properties which are similar to those of the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \quad \text{Re } s > 1, \quad (1.85)$$

where the product on the r.h.s. extends over all primes  $p$ , see [27] for a detailed discussion.  $\zeta(s)$  can be extended to a meromorphic function with one simple pole at  $s = 1$  and zeros on the real axis at  $s = -2, -4, -6, \dots$ . The infinitely many complex zeros are located in the strip  $0 < \text{Re } s < 1$ ; according to the famous Riemann hypothesis these are of the form  $s_n = \frac{1}{2} \pm it_n$ ,  $t_n > 0$ . One can hence introduce the counting function

$$N_R(T) := \# \{n; 0 \leq t_n \leq T\} = \overline{N}_R(T) + \frac{1}{\pi} \arg \zeta \left( \frac{1}{2} + iT \right), \quad (1.86)$$

where  $\overline{N}_R(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O(\frac{1}{T})$  for  $T \rightarrow \infty$ . The Riemann zeta function also satisfies a functional equation,

$$\zeta(1-s) = \frac{2^{-s} \pi^{1-s}}{\Gamma(1-s) \sin(\frac{\pi s}{2})} \zeta(s), \quad (1.87)$$

so that

$$\xi_R(t) := \zeta \left( \frac{1}{2} - it \right) e^{-2\pi i \overline{N}_R(t)} \quad (1.88)$$

is real and even for  $t \in \mathbb{R}$ ,  $\xi_R(-t) = \xi_R(t)$ .

The close similarity between the Riemann zeta function  $\zeta(s)$  and the Selberg zeta function  $Z(s)$ , compare in particular (1.83) and (1.86), suggests to view the complex zeros  $s_n = \frac{1}{2} \pm it_n$  as analogues of the spectral zeros  $s_n = \frac{1}{2} \pm ip_n$  of  $Z(s)$ . Thus the quantities  $t_n^2 + \frac{1}{4}$  might serve as ‘model quantum energies’ and their distribution might imitate the distribution of the quantum energies  $E_n = p_n^2 + \frac{1}{4}$ .

## 2 Eigenvalue Statistics and Periodic Orbit Theory

As already mentioned in the Introduction, one of the basic assumptions in the field of quantum chaos is that one should be able to observe fingerprints of classical chaos in the distribution of quantum energies. That is, given the sequence

$$0 \leq E_1 \leq E_2 \leq E_3 \leq \dots \quad (2.1)$$

of eigenvalues of some quantum Hamiltonian  $\hat{H}$ , the distribution of the quantum energies is expected to follow certain universal rules which allow to identify whether the corresponding classical dynamics are chaotic or regular. In order to investigate statistical properties of a spectrum suitable quantities that reflect the distribution of eigenvalues are required. Obviously, the spectral staircase

$$N(E) := \# \{n; E_n \leq E\} = \int_0^E d(E') \, dE' \quad (2.2)$$

encodes all information on spectral statistics and thus turns out to be a fundamental object for our further purposes. Due to the trace formula (1.38) for the spectral density  $d(E)$ ,

$$N(E) = \overline{N}(E) + N_{fl}(E) \quad (2.3)$$

arises as a natural decomposition of the staircase function into a part  $\overline{N}(E)$  that derives from the contribution (1.21) of  $t = 0$  to the trace formula, and a further contribution  $N_{fl}(E)$  reflecting the effect of the non-trivial classical periodic orbits. When discussing the Selberg trace formula we noticed that for certain classically chaotic systems the contribution of the classical periodic orbits to the spectral staircase is contained in the phase of the Selberg zeta function via  $N_{fl}(E) = \frac{1}{\pi} \arg Z(\frac{1}{2} + ip)$  with  $E = p^2 + \frac{1}{4}$ , see (1.83). Since  $\overline{N}(E)$  essentially gives the volume of that part of the classical phase space where  $H(p, x) \leq E$ , see (1.2), the interesting details of the spectral statistics are contained in  $N_{fl}(E)$ ; the latter describes the fluctuations of the staircase about its mean behaviour  $\overline{N}(E)$ . It therefore proves useful to defold the spectrum, i.e., to introduce the scaled eigenvalues  $x_n := \overline{N}(E_n)$ . By abuse of notation we now denote the counting function for the  $x_n$ 's by

$$N(x) := \# \{n; x_n \leq x\} = x + N_{fl}(x) . \quad (2.4)$$

The corresponding spectral density then reads

$$d(x) = \sum_n \delta(x - x_n) = 1 + d_{fl}(x) , \quad (2.5)$$

so that the scaled eigenvalues  $x_n$  have a unit mean separation. Defolded spectra can thus be directly compared in their statistical properties.

## 2.1 Measures for the Distribution of Eigenvalues

The basic idea behind spectral statistics is to view the list  $x_1, x_2, x_3, \dots$  of scaled eigenvalues of a quantum Hamiltonian as a sequence of random events. The spectral staircase function (2.2) is then considered as a sample function of a random process. The ultimate goal therefore is to characterize and classify all possible such random processes in terms of properties of the –classical and/or quantum mechanical– physical systems giving rise to the defolded spectra.

At first one might wonder what can be random about a spectrum of a fixed quantum Hamiltonian? In order to answer this question one should recall the mathematical concept of probability. In general, the starting point is a probability space, i.e., a measure space with a normalized measure. Any measurable real valued function on a probability space is then considered as a random variable. Now,  $N(x)$  obviously is an integer valued and piecewise continuous function on the positive real line  $\mathbb{R}_+ = (0, \infty)$ . In order to turn the latter into a probability space one needs to define a normalized measure. Since  $\mathbb{R}_+$  is unbounded one cannot simply use Lebesgue measure because it is not finite. Instead, one restricts attention to a bounded region on the positive real line, say to an interval  $I_x := [x - \Delta x, x + \Delta x]$  of width  $2\Delta x$  that is centered at  $x$ . In order to obtain information on the asymptotic distribution of eigenvalues one then has to consider the limit  $x \rightarrow \infty$ . As this is equivalent to the semiclassical limit  $\hbar \rightarrow 0$  the asymptotic analysis of spectral statistics allows for an application of semiclassical methods as, e.g., the use of trace formulae.

In defolded spectra the mean separation of eigenvalues is one so that in the mean the intervals  $I_x$  contain  $2\Delta x$  quantum energies. The latter number should approach infinity in order to yield reasonable limit distributions as  $x \rightarrow \infty$ . We therefore choose  $\Delta x = c x^\alpha$ , with some suitable constants  $c > 0$  and  $\alpha$ . If not stated otherwise,  $\alpha$  will be fixed to obey  $\frac{1}{2} < \alpha < 1$ . In view of the limit  $x \rightarrow \infty$  we intend to perform, the unboundedness of the spectral staircase suggests an alternative choice for the random process to be studied as a means to characterize spectral statistics. One customarily introduces the number

$$n_L(y) := N(y + L) - N(y) , \quad y \in I_x , \quad (2.6)$$

of eigenvalues in intervals  $(y, y + L]$  of length  $L > 0$ . On the interval  $I_x$ , from which  $y$  has to be chosen randomly, we make the simplest choice for a probability measure,  $d\mu_x(y) := \frac{1}{2\Delta x} \chi_{I_x}(y) dy$ , where  $\chi_{I_x}(y)$  denotes the characteristic function of the interval  $I_x$ . Given an integrable function  $f(y)$

on  $I_x$  its mean value with respect to  $d\mu_x(y)$  is thus given by

$$\mathbb{E}_x[f] := \int f(y) d\mu_x(y) = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} f(y) dy . \quad (2.7)$$

One can choose as a probability measure on  $I_x$  any normalized measure of the form  $w(y) dy$ , where  $w(y) \geq 0$  is continuous and vanishes outside  $I_x$ . The limit distributions then induced by the random variable  $n_L(y)$  on  $I_x$ , for  $x \rightarrow \infty$ , do not depend on the choice of  $w(y)$ , see [28] for a discussion on this point. We therefore stick to our favourite choice  $d\mu_x(y)$ .

At this point we introduce some notation that will be kept throughout the rest of this presentation: Above we indicate the interval  $I_x$  by a subscript  $x$  which we attach to the random variables and their distributions. When passing to the limit  $x \rightarrow \infty$ , the resulting distributions take care of the asymptotic properties of a given spectrum. All quantities corresponding to this limit will then carry no subscript. The same notation will be used in case a random process is stationary, i.e., no dependence on  $x$  occurs anyway.

Now, for each value of the parameter  $L$  the function  $n_L(y)$  is a random variable on the interval  $I_x$ , and thus upon varying  $L$  it defines a continuous parameter random process. The ultimate aim then is to investigate the resulting limit random process as  $x \rightarrow \infty$  as far as possible. On the way, however, we will notice that already at finite  $x$  one can obtain interesting information. As long as  $L$  is finite,  $n_L(y)$  provides information on correlations among approximately  $L$  eigenvalues and therefore yields the local spectral statistics. In order to reach the global scale one has to perform the limit  $L \rightarrow \infty$ . Starting with  $n_L(y)$  on  $I_x$  one is therefore confronted with a competition of the two limits  $x \rightarrow \infty$  and  $L \rightarrow \infty$ . Phenomena occurring in this context are the central topic of the third part of these lectures.

For the further analysis it proves useful to introduce the probability  $E_x(k; L)$  to observe a value of  $k$  for the random variable  $n_L(y)$  when  $y$  is uniformly distributed on the interval  $I_x$ ,

$$E_x(k; L) := \mathbb{E}_x[\delta_{n_L, k}] = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \delta_{n_L(y), k} dy , \quad (2.8)$$

where  $\delta_{k, l}$  denotes Kronecker's symbol. According to (2.5) and (2.7) the mean value of  $n_L(y)$  reads

$$\mathbb{E}_x[n_L] = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} [N(y+L) - N(y)] dy = L + \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} [N_{fl}(y+L) - N_{fl}(y)] dy . \quad (2.9)$$

A simple result emerges in the limit  $x \rightarrow \infty$  because  $N_{fl}(y)$  describes the fluctuations of the spectral staircase about its mean behaviour  $\bar{N}(y) = y$ . Thus in particular

$$\lim_{x \rightarrow \infty} \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} N_{fl}(y) dy = 0 , \quad (2.10)$$

so that the mean value  $\mathbb{E}_x[n_L]$  asymptotically approaches  $L$ . See [29] for a discussion of (2.10) which makes use of the trace formula. Having in mind the limit  $x \rightarrow \infty$  as our actual goal, we introduce the  $m$ -th moments of the shifted random variable  $n_L(y) - L$ ,

$$\Sigma_x^m(L) := \mathbb{E}_x[(n_L - L)^m] = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} [N(y+L) - N(y) - L]^m dy . \quad (2.11)$$

Due to the above remark  $\Sigma_x^1(L)$  vanishes as  $x \rightarrow \infty$  so that the lowest non-trivial moment is given by the so-called number variance

$$\Sigma_x^2(L) = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} [N(y+L) - N(y) - L]^2 dy = \sum_{k=0}^{\infty} (k-L)^2 E_x(k; L) . \quad (2.12)$$

One can easily determine the behaviour of the number variance for small  $L$ : Since  $I_x$  contains finitely many scaled eigenvalues there exists a positive minimal separation  $s_n := x_{n+1} - x_n$  of neighbouring levels. Once  $L$  is smaller than the minimal  $s_n$  every interval  $(y, y + L]$  with  $y \in I_x$  either contains one or no  $x_n$ , and thus  $E_x(k; L) = 0$  for  $k \geq 2$ . One then employs the normalization of the probabilities,  $E_x(0; L) + E_x(1; L) = 1$ , to observe

$$\Sigma_x^2(L) = \sum_{k=0}^1 (k - L)^2 E_x(k; L) = (1 - 2L) \mathbb{E}_x[n_L] + L^2 . \quad (2.13)$$

The relation (2.9) then implies that

$$\Sigma_x^2(L) \sim L , \quad L \rightarrow 0 . \quad (2.14)$$

The behaviour of  $\Sigma_x^2(L)$  for small  $L$  does not provide detailed information on the distribution of eigenvalues since for too small  $L$  the number variance merely reflects the fact that  $N(x)$  is a staircase function.

The opposite limit  $L \rightarrow \infty$  cannot so easily be dealt with, but on the other hand reflects specific properties of the underlying physical system. From (2.12) we obtain

$$\begin{aligned} \Sigma_x^2(L) &= \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} [N(y+L) - N(y) - L]^2 dy \\ &= \mathbb{E}_{x+L} [N_{fl}^2] + \mathbb{E}_x [N_{fl}^2] - 2\mathbb{E}_x [N_{fl}(y+L)N_{fl}(y)] . \end{aligned} \quad (2.15)$$

Certainly, the limit  $L \rightarrow \infty$  also requires to perform  $x \rightarrow \infty$ , since otherwise the intervals  $(y, y + L]$  for  $y \in I_x$  hardly overlap with  $I_x$ ; by the same reason  $L$  should not grow faster than  $x$ . We therefore choose  $L = L(x) = a x^\gamma$  with some positive constants  $a$  and  $\gamma \leq 1$ . Now two effects on the behaviour of  $\Sigma_x^2(L)$  in this limit are to be observed on the r.h.s. of (2.15). On the one hand, a large exponent  $\gamma$  will decouple  $N_{fl}(y+L)$  and  $N_{fl}(y)$  to a certain extent so that the contribution of the correlation function on the r.h.s. of (2.15) becomes smaller in absolute value. On the other hand, a small exponent  $\gamma$  will turn the two expectation values of  $N_{fl}^2$  almost equal. However, in the asymptotic regime for  $x \rightarrow \infty$ ,  $\gamma$  need not really be small because once  $\mathbb{E}_x[N_{fl}^2]$  increases for  $x \rightarrow \infty$ , then  $\mathbb{E}_{x+L}[N_{fl}^2] \sim \mathbb{E}_x[N_{fl}^2]$  for any choice  $L = a x^\gamma$  with  $\gamma < 1$ . It indeed turns out that in typical cases the expectation values of  $N_{fl}^2(y)$  on  $I_x$  behave either like

$$\mathbb{E}_x [N_{fl}^2] \sim b_1 \log x \quad \text{or} \quad \mathbb{E}_x [N_{fl}^2] \sim b_2 x^\rho , \quad (2.16)$$

with some  $\rho > 0$ . We hence expect that as long as  $\gamma$  is larger than some critical exponent  $\gamma_c$  the number variance is asymptotically given by the dominant and  $L$ -independent contribution  $2 \mathbb{E}_x [N_{fl}^2]$ , with  $L$ -dependent modifications provided by the correlation function  $\mathbb{E}_x [N_{fl}(y+L)N_{fl}(y)]$ . Theoretical [30] as well as numerical [30, 31] studies indeed confirm this qualitative picture. In section 2.3 we will return to a more detailed discussion of this point.

The characteristic function of a probability distribution is defined as the Fourier transform of the probability measure. In the case of the random variable  $n_L(y)$  it is given by

$$J_{n_L, x}(\xi) := \mathbb{E}_x [e^{i\xi n_L}] = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} e^{i\xi n_L(y)} dy . \quad (2.17)$$

The probabilities  $E_x(k; L)$ , see (2.8), can be recovered from the characteristic function because

$$\begin{aligned} E_x(k; L) &= \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \delta_{n_L(y), k} dy = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i\xi [n_L(y) - k]} d\xi dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-i\xi k} J_{n_L, x}(\xi) d\xi . \end{aligned} \quad (2.18)$$

Hence the knowledge of the characteristic function is equivalent to knowing the probabilities  $E_x(k; L)$  themselves. Characteristic functions can be used to test the statistical independence of random variables since the characteristic function of the sum of independent random variables factorizes into the product of the respective characteristic functions of the individual random variables. In our case,  $n_L(y) = L + N_{fl}(y + L) - N_{fl}(y)$  essentially is the sum of the two random variables  $N_{fl}(y + L)$  and  $-N_{fl}(y)$ , with  $y \in I_x$ . If one wants to test their statistical independence one therefore has to study whether or not the characteristic function (2.17) factorizes. Certainly, a complete statistical independence can only be expected for the limit distributions as  $x \rightarrow \infty$ , since only these take infinitely many eigenvalues into account. As argued above, a decoupling of the distributions of  $N_{fl}(y + L)$  and  $N_{fl}(y)$  will then require to perform also  $L \rightarrow \infty$ , with  $L = ax^\gamma$  and a positive exponent  $\gamma$  which is large enough. One way to determine the minimal such  $\gamma$ , which then yields  $\gamma_c$ , could be to test the factorization of the characteristic function  $J_{n_L, x}(\xi)$ . At the end of section 2.3 we will return to this topic in some more detail.

## 2.2 Random Matrix Theory

So far we discussed the distribution of eigenvalues of individual quantum Hamiltonians, without specifying any result about, e.g., the probabilities  $E_x(k; L)$ . However, we already addressed the expectation to observe fingerprints of the classical dynamics on the corresponding distribution of quantum energies. Ideally, this could mean a universal behaviour of the probabilities  $E_x(k; L)$ , as  $x \rightarrow \infty$  of course, depending only on the type of the classical dynamics. For example, one could expect that all classically chaotic systems lead to one and the same limit distribution for their respective random variables  $n_L$ ; classically integrable systems then should produce a distinctively different distribution. This expectation being very strong could be weakened in replacing ‘all systems’ by ‘almost all systems’ in order to account for ‘special’ or ‘non-generic’ cases where exceptions might occur. Of course, one then has to specify what ‘almost all’ means in this context. To this end it is necessary to consider a family of quantum Hamiltonians  $\hat{H}_\lambda$ , where  $\lambda$  ranges over some parameter space  $\Lambda$ . Since now one is dealing with families  $\{E_1(\lambda), E_2(\lambda), E_3(\lambda), \dots\}$  of quantum spectra all spectral quantities depend on the parameter  $\lambda$ , e.g.,  $n_{L, \lambda}(y)$ ,  $E_{x, \lambda}(k; L)$ . Provided the parameter space  $\Lambda$  is endowed with a suitable measure  $d\nu(\lambda)$ , a weak form of universality in spectral statistics would be reflected in the fact that the limit probabilities  $E_\lambda(k; L)$ , arising from  $E_{x, \lambda}(k; L)$  as  $x \rightarrow \infty$ , are given by  $E_\Lambda(k; L)$  for almost all  $\lambda \in \Lambda$  with respect to the measure  $d\nu(\lambda)$ . If the latter were normalized,  $\int_\Lambda d\nu(\lambda) = 1$ , i.e., it were a probability measure, then one would obtain

$$E_\Lambda(k; L) = \int_\Lambda E_\lambda(k; L) d\nu(\lambda) = \int_\Lambda E_\lambda(k; L) d\nu(\lambda) . \quad (2.19)$$

Thus, even in case it is not known whether  $E_\lambda(k; L) = E_\Lambda(k; L)$  for almost all  $\lambda \in \Lambda$  it would be interesting to calculate the mean value on the r.h.s. of (2.19) which describes the spectral statistics of the quantum Hamiltonians  $\hat{H}_\lambda$  on average. In this perspective the limit of  $E_{x, \lambda}(k; L)$  as  $x \rightarrow \infty$  may even not exist for a set of  $\lambda$  which is of zero measure in  $\Lambda$ .

Examples for families of classically chaotic systems to which the above programme might be applied are provided by geodesic motions on surfaces of constant negative curvature as discussed in section 1.3. The compact such surfaces with the topology of a sphere with  $g$  handles ( $g \geq 2$ ) form a  $(6g - 6)$ -parameter family. For  $g = 2$  Aurich and Steiner [32] calculated numerically eigenvalues of quantum Hamiltonians on 30 surfaces which they picked out randomly from the corresponding 6-parameter family, and computed averages of the probabilities  $E_{x, \lambda}(k; L)$  over their sample of 30 values for the parameter  $\lambda$ . Apart from the fact that  $x$  necessarily had to be kept finite and thus the saturation effects to be discussed in section 2.3 showed up, it was observed that the ‘ensemble averages’ agreed with the respective probabilities in the Gaussian orthogonal ensemble (GOE) of random matrix theory. We remark that in this example it turns out to be essential not to expect the GOE result for all values of  $\lambda$  since certain exceptions, which go under the notion of arithmetic quantum chaos [33, 34, 35], are

known to exist. However, with respect to any reasonable measure  $d\nu(\lambda)$  these will form a set of zero measure.

In a purely mathematical model the programme has recently been completely carried out by Katz and Sarnak [37]. As model operators they consider matrices from the classical compact Lie groups  $O(N)$ ,  $SO(N)$ ,  $U(N)$ ,  $SU(N)$ , and  $USP(N)$ . Then the respective group itself can serve as the parameter space  $\Lambda$ , and the most natural choice for  $d\nu(\lambda)$  is provided by the Haar measure on the group. Katz and Sarnak prove that in the limit  $N \rightarrow \infty$  almost all matrices from any one of the classical groups show the spectral statistics of the Gaussian unitary ensemble (GUE) of random matrix theory.

In random matrix theory, see [38] and Bohigas' contribution to these proceedings, one attempts to construct from certain assumptions the probabilities  $E_\Lambda(k; L)$  which are expected for 'almost all' quantum Hamiltonians of a given type. In this context the first step consists of cutting off the dimension of the Hilbert space on which a Hamiltonian  $\hat{H}$  is defined at a finite value  $N$ . Then, upon choosing some orthonormal basis (onb) in the Hilbert space,  $\hat{H}$  is represented by a hermitian  $N \times N$ -matrix  $\mathbb{H}$ . In case the physical system from which  $\mathbb{H}$  derives is time-reversal invariant one can always choose the onb in such a way that  $\mathbb{H}$  is real symmetric. For simplicity we will in the following always focus our attention to this case. One then still has the freedom to choose another onb without changing  $\hat{H}$  and its spectral properties; only the matrix representation is affected. The base changes in the  $N$ -dimensional Hilbert space are provided by orthogonal transformations  $O \in O(N)$ . The matrix  $\mathbb{H}$  is thus conjugated to  $O^{-1}\mathbb{H}O$ . As a model space of truncated Hamiltonians one therefore considers the set of all real symmetric matrices  $\mathbb{H}$ , with an identification of  $\mathbb{H}$  and  $O^{-1}\mathbb{H}O$  whenever  $O \in O(N)$ . Altogether a real symmetric  $N \times N$ -matrix has  $\frac{1}{2}N(N+1)$  independent entries. The symmetry group  $O(N)$  is of dimension  $\frac{1}{2}N(N-1)$  so that the set of classes of equivalent real symmetric matrices has dimension  $N$ . A possible choice for coordinates of the space of truncated Hamiltonians is obtained after diagonalizing each  $\mathbb{H}$ : There exists an  $O \in O(N)$  such that  $O^{-1}\mathbb{H}O$  is diagonal. Therefore each  $\mathbb{H}$  is equivalent to a diagonal matrix, with its eigenvalues  $E_1, \dots, E_N$  as entries on the diagonal. The eigenvalues then yield  $N$  parameters to describe  $\mathbb{H}$ , and the additional  $\frac{1}{2}N(N-1)$  coordinates required to characterize  $\mathbb{H}$  completely are provided by the independent entries of  $O$ .

The basic assumption of random matrix theory now concerns the measure  $d\nu_N(\mathbb{H})$  on the space of real symmetric  $N \times N$ -matrices. It is based on the principle of minimal knowledge and consists of two parts: (i)  $d\nu_N(\mathbb{H}) = P_N(\mathbb{H}) d\mathbb{H}$ , where  $d\mathbb{H} = \prod_{k \leq l} d\mathbb{H}_{kl}$ , and  $P_N(\mathbb{H}) = P_N(O^{-1}\mathbb{H}O)$ , so that the measure does not depend on a particular choice of an onb. (ii) The independent entries  $\mathbb{H}_{kl}$ ,  $k \leq l$ , are statistically independent. By these requirements the measure  $d\nu_N(\mathbb{H})$  is already fixed to read [38]

$$d\nu_N(\mathbb{H}) = \text{const.} \exp \{ -a \text{tr } \mathbb{H}^2 + b \text{tr } \mathbb{H} \} d\mathbb{H} , \quad (2.20)$$

with some real constants  $a > 0$  and  $b$ . In order to obtain a more explicit expression one changes variables from  $\mathbb{H}_{kl}$ ,  $k \leq l$ , to  $E_1, \dots, E_N$  and  $\frac{1}{2}N(N-1)$  further variables which parametrize the orthogonal matrix  $O$  that diagonalizes  $\mathbb{H}$ . One finally observes [38]

$$d\nu_N(\mathbb{H}) = d\rho(O) \prod_{k < l} |E_k - E_l| \prod_{j=1}^N e^{-aE_j^2} dE_j , \quad (2.21)$$

where  $d\rho(O)$  denotes some measure that only depends on  $O$ . Upon integrating out  $d\rho(O)$  one has thus obtained a measure on the model space of truncated Hamiltonians, where the latter are being parametrized by their eigenvalues. Notice that the eigenvalues are not statistically independent since the measure does not factorize. The term  $\prod_{k < l} |E_k - E_l|$  rather introduces a correlation of the eigenvalues that makes small separations  $E_{k+1} - E_k$  of neighbouring eigenvalues less probable. This effect is known as level repulsion.

The limit measure resulting from (2.21) as  $N \rightarrow \infty$  is commonly called Gaussian orthogonal ensemble (GOE). Given the latter one can then calculate all statistical properties of eigenvalues of



random symmetric matrices. For example, the number variance reads

$$\begin{aligned}\Sigma_{GOE}^2(L) &= \frac{2}{\pi^2} [\log(2\pi L) + \gamma + 1 - \cos(2\pi L) - \text{Ci}(2\pi L)] \\ &\quad + 2L \left[ 1 - \frac{2}{\pi} \text{Si}(2\pi L) \right] + \frac{1}{\pi^2} \text{Si}(\pi L)^2 - \frac{1}{\pi} \text{Si}(\pi L) \\ &= \frac{2}{\pi^2} \left[ \log(2\pi L) + \gamma + 1 - \frac{\pi^2}{8} \right] + O\left(\frac{1}{L}\right),\end{aligned}\tag{2.22}$$

where

$$\text{Si}(x) := \int_0^x \frac{\sin y}{y} dy \quad \text{and} \quad \text{Ci}(x) := \gamma + \log x + \int_0^x \frac{\cos y - 1}{y} dy.\tag{2.23}$$

Here  $\gamma = 0.577215\dots$  denotes Euler's constant. For the GOE one does not know, however, an explicit closed expression for the probabilities  $E_{GOE}(k; L)$  to find  $k$  defolded eigenvalues in spectral intervals of length  $L$ ; but these quantities can be calculated numerically to a high precision. That way Aurich and Steiner [32] found

$$E_{GOE}(k; L) \sim \frac{1}{\sqrt{2\pi\Sigma_{GOE}^2(L)}} e^{-\frac{(k-L)^2}{2\Sigma_{GOE}^2(L)}}, \quad L \rightarrow \infty.\tag{2.24}$$

Later Costin and Lebowitz [39] made precise the way the limit of large  $L$  has to be understood. Since both the mean value  $\mathbb{E}_{GOE}[n_L] = L$  and the variance (2.22) of the random variable  $n_L$ , when considered in the GOE, diverge as  $L \rightarrow \infty$  one has to employ a renormalization before one performs the limit. Costin and Lebowitz proved that after renormalization the distribution of the random variable obtained from  $n_L$  weakly approaches a standard Gaussian, see section 3.1 for a further discussion.

The construction of the GOE relies on a number of assumptions that cannot be derived from first principles when one is discussing the distribution of eigenvalues of actual quantum Hamiltonians. Historically, random matrix theory was introduced to describe the statistics of energy levels and of resonances of large atomic nuclei. In view of the huge number of degrees of freedom involved the principle of minimal knowledge seemed to be reasonable in this context. It indeed turned out to result in a correct description of experimental data, see [40] for a collection of the original contributions. Later Bohigas, Giannoni, and Schmit [41] conjectured that the eigenvalue statistics of quantum Hamiltonians with even a low number ( $\geq 2$ ) of degrees of freedom universally follow the predictions of random matrix theory, if only the corresponding classical dynamics were chaotic. Again, the conjecture found confirmation by numerical calculations of eigenvalues and their statistics in many examples.

For classically integrable systems the situation appears to be completely different. Since at least a second classical integral of motion exists one has a quantum mechanical observable at hand that commutes with the Hamiltonian. For each value of the quantum number that corresponds to the second conserved quantity one obtains a subspectrum of the quantum energy spectrum. It now seems that the entire energy spectrum behaves like an independent superposition of these, typically infinitely many, subspectra. Provided the superposition of subspectra were indeed uncorrelated, the complete spectrum would have the local statistics of a Poissonian random process, i.e.,

$$E_{Poisson}(k; L) = \frac{L^k}{k!} e^{-L}.\tag{2.25}$$

In particular, the density  $P(s)$  of the distribution of nearest neighbour level spacings  $s_n := x_{n+1} - x_n$  of defolded eigenvalues would be exponential,  $P_{Poisson}(s) = e^{-s}$ . Based on a semiclassical analysis the Poissonian behaviour of the local eigenvalue statistics was predicted by Berry and Tabor [42] and meanwhile has been verified by numerical calculations in many examples. See e.g. [43] for a careful numerical analysis.

But again, exceptions are known to exist. In order to give an example suppose that  $e_1, e_2 \in \mathbb{R}^2$  are linearly independent such that  $\mathcal{L} = \mathbb{Z}e_1 + \mathbb{Z}e_2$  is a lattice in  $\mathbb{R}^2$ . Then  $\mathbb{R}^2/\mathcal{L}$  is a two dimensional flat torus. The classical geodesic motion on  $\mathbb{R}^2/\mathcal{L}$  is integrable, and the spectrum of  $\hat{H} = -\frac{\hbar^2}{2m}\Delta$  on  $L^2(\mathbb{R}^2/\mathcal{L})$  is explicitly known; it comprises of the eigenvalues  $E_{k,l} = \frac{\hbar^2}{2m}4\pi^2(kf_1 + lf_2)^2$ . Here  $k$  and  $l$  run through  $\mathbb{Z}$ , and  $\{f_1, f_2\}$  is a basis for the dual lattice  $\mathcal{L}^*$  of  $\mathcal{L}$ , i.e.,  $e_i \cdot f_j = \delta_{ij}$ . Hence

$$E_{k,l} = \frac{\hbar^2}{2m} Q(k, l) = \frac{\hbar^2}{2m} [A k^2 + B kl + C l^2] , \quad (2.26)$$

where  $A, B, C$  are suitable real constants which determine  $Q(k, l)$  to be a positive definite binary quadratic form. If  $\frac{B}{A}$  and  $\frac{C}{A}$  are rational numbers,  $Q(k, l)$  is a rational quadratic form. In this case the value of  $Q(k, l)$  itself is a rational number for all  $k, l \in \mathbb{Z}$ ; it can in particular be an integer,  $Q(k, l) = n$ ,  $n \in \mathbb{N}$ . If we denote the number of pairs  $(k, l) \in \mathbb{Z}^2$  that represent  $n$  as  $n = Q(k, l)$  by  $r_Q(n)$ , the multiplicity of the eigenvalue  $E = \frac{\hbar^2}{2m} n$  is  $r_Q(n)$ . Now, for rational forms it is known that  $r_Q(n)$  is unbounded as  $n \rightarrow \infty$ . One can then conclude that the distribution of level spacings becomes singular at  $s = 0$ ; it is actually given by the ‘density’  $P(s) = \delta(s)$ , see for example [44].

The coefficients  $A, B, C$  in (2.26) provide parameters for the set  $\Lambda$  of all quantum Hamiltonians  $\hat{H}_\lambda$  that derive from Laplacians on flat tori. The positivity of the eigenvalues only requires the discriminant  $D_Q = B^2 - 4AC$  to be negative. The set of Hamiltonians in  $\Lambda$  corresponding to rational quadratic forms then clearly is of zero Lebesgue measure, so that the tori leading to the singular distribution of level spacings as described above are non-generic in a well defined sense. It is widely believed, however not known, that for a set of full Lebesgue measure in  $\Lambda$  the level spacings distribution has an exponential density,  $P(s) = e^{-s}$ . As an important step towards a proof of Poissonian local eigenvalue statistics for Laplacians on flat tori Sarnak recently proved [44] that the spectral statistics, when measured with the pair correlation function, is Poissonian on a set of full Lebesgue measure in  $\Lambda$ . Moreover, if  $B = 0$  and  $\frac{C}{A}$  satisfies a certain Diophantine condition, Bleher and Lebowitz [45] proved that the number variance  $\Sigma^2(L)$  is Poissonian. In view of the above discussion a further result of Sarnak is rather amazing. He could show that there exists a set  $\tilde{\Lambda} \subset \Lambda$  of tori, which is generic in a topological sense, and on which the level spacings distribution has no exponential density; in fact  $\int_0^{1/3} P_\lambda(s) ds$  does even not exist for  $\lambda \in \tilde{\Lambda}$ . The set  $\tilde{\Lambda}$  is generic in the sense that it is of second Baire category, i.e., it is a countable intersection of open dense sets. The lesson to learn from this example is that universality in spectral statistics can only hold in a weak form, as for example expressed around (2.19). One even has to be very careful in the way one is declaring particular quantum Hamiltonians as ‘generic’.

### 2.3 A Semiclassical Analysis of Spectral Statistics

It seems that the only tools available to investigate the distribution of eigenvalues of a single quantum Hamiltonian  $\hat{H}$  are provided by semiclassical trace formulae. These relate the spectrum of  $\hat{H}$  to classical quantities: periodic orbits and their stabilities and Maslov phases. One therefore might anticipate that the asymptotic distribution of eigenvalues is completely determined by the corresponding classical dynamics. The first to perform such a semiclassical analysis, to classically integrable systems, were Berry and Tabor [42]. Later, Berry extended this analysis to classically chaotic systems as well, see [46, 30] and his contribution to [3]. Here we only want to consider quantum systems with a strongly chaotic classical limit, which shall in particular mean that all classical periodic orbits be unstable and isolated in phase space. To be definite, we further restrict our attention to cases where the trace formula (1.48) applies, and moreover, the spectral density has an asymptotic expansion as in (1.43). Thus, the classical dynamics are given by either billiards or geodesic motions on surfaces. The case of the Selberg trace formula (1.71) is therefore included.

The starting point for our discussion is the representation

$$n_L(y) = N(y+L) - N(y) = \int_y^{y+L} d(u) du = L + \int_y^{y+L} d_{fl}(u) du \quad (2.27)$$

of the random variable  $n_L(y)$ ,  $y \in I_x$ , in terms of  $d_{fl}(u)$ , compare (2.5). Now, the fluctuating part of the spectral density, when expressed in terms of the momentum variable  $p$ , can be given in terms of a periodic orbit sum as in (1.59). The number variance, being the second moment of  $n_L(y) - L$ , can hence be represented as

$$\begin{aligned} \Sigma_x^2(L) &= \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} [n_L(y) - L]^2 dy \\ &= \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \int_y^{y+L} \int_y^{y+L} d_{fl}(u_1) d_{fl}(u_2) du_1 du_2 dy. \end{aligned} \quad (2.28)$$

To proceed further, one customarily introduces the two-level form factor

$$K_x(\tau) := \int_{-\infty}^{+\infty} e^{2\pi i t \tau} \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} d_{fl}(y) d_{fl}(y+t) dy dt. \quad (2.29)$$

By Fourier inversion and a change of variables one obtains from (2.28) and (2.29) that

$$\Sigma_x^2(L) = \int_{-\infty}^{+\infty} d\tau \int_0^L d\rho_1 \int_0^L d\rho_2 e^{-2\pi i \tau(\rho_1 - \rho_2)} K_{x+\rho_1}(\tau). \quad (2.30)$$

As we are finally interested in the limit of  $x \rightarrow \infty$ , with  $0 \leq \rho_1 \leq L \ll x$ , we approximate  $K_{x+\rho_1}(\tau) \sim K_x(\tau)$  and observe

$$\Sigma_x^2(L) \sim \frac{2}{\pi^2} \int_0^\infty \frac{\sin^2(\pi L \tau)}{\tau^2} K_x(\tau) d\tau. \quad (2.31)$$

After having performed  $x \rightarrow \infty$  the above asymptotic relation indeed turns into an identity. When differentiating the latter with respect to  $L$  and applying a sine-Fourier inversion afterwards, one obtains [31]

$$K(\tau) = \frac{d}{dL} \Sigma^2(L) \Big|_{L=0} - \frac{1}{2\pi\tau} \int_0^\infty \sin(2\pi L \tau) \frac{d^3}{dL^3} \Sigma^2(L) dL, \quad (2.32)$$

from which one concludes that

$$K(\tau) = 1 + O\left(\frac{1}{\tau}\right), \quad \tau \rightarrow \infty, \quad (2.33)$$

since  $\Sigma^2(L) \sim L$  for  $L \rightarrow 0$ , see (2.14).

We are now going to express the form factor in terms of sums over classical periodic orbits. To this end  $d_{fl}(x)$  in (2.29) has to be represented by the periodic orbit sum (1.59). However, since the form factor (2.29) is given in terms of the defolded spectral variable  $x = \overline{N}(E)$ , we first have to rewrite it as a function of the energy  $E = p^2$ . From the relation (1.43) we conclude that  $\overline{N}(E) \sim \frac{A}{4\pi\hbar^2} E$  as  $E \rightarrow \infty$  or  $\hbar \rightarrow 0$ , and thus  $\overline{d}(E) \sim \frac{A}{4\pi\hbar^2} =: \overline{d}$ . We therefore observe that

$$d_{fl}(x) = \frac{d}{dx} N_{fl}(x) = \frac{dE}{dx} \frac{dN_{fl}(E)}{dE} = \frac{d_{fl}(E)}{\overline{d}(E)} \sim \overline{d}^{-1} d_{fl}(E). \quad (2.34)$$

The two-level form factor, when expressed by  $E$  instead of  $x$ , hence reads

$$K(\tau; E) \sim \overline{d}^{-1} \int_{-\infty}^{+\infty} e^{2\pi i \tau \overline{d} \lambda} \frac{1}{2\Delta E} \int_{E-\Delta E}^{E+\Delta E} d_{fl}\left(E' - \frac{\lambda}{2}\right) d_{fl}\left(E' + \frac{\lambda}{2}\right) dE' d\lambda, \quad (2.35)$$

where we now correlate  $d_{fl}$  at the symmetrically placed points  $E' \pm \frac{\lambda}{2}$  instead of  $E'$  and  $E' + \lambda$ .

We are now in a position to employ the semiclassical periodic orbit sum (1.59) for  $\tilde{d}_{fl}(p) = 2p d_{fl}(E)$ ,  $p = \sqrt{E}$ , in (2.35). The result, which was first obtained by Berry [46], then reads

$$K(\tau; E) \sim \frac{4p^2}{l_H^2} \left\langle \sum_{l_n, l_m} \sum_{r, s \neq 0} A_{n,r} A_{m,s} e^{\frac{i}{\hbar} p' (r l_n - s l_m)} \delta \left( \tau - \frac{r l_n + s l_m}{2 l_H} \right) \right\rangle, \quad (2.36)$$

where

$$A_{n,r} = \frac{g_n l_n}{2p} \frac{e^{-i \frac{\pi}{2} r \tilde{\mu}_n}}{|\det(M_n^r - \mathbf{1})|^{\frac{1}{2}}} \sim \frac{g_n l_n}{2p} e^{-i \frac{\pi}{2} r \tilde{\mu}_n} e^{-\frac{r}{2} u_n}, \quad l_n \rightarrow \infty, \quad (2.37)$$

is an amplitude attached to the  $r$ -fold repetition of the primitive length  $l_n$ , which is of multiplicity  $g_n$ . The inner double sum in (2.36) extends over all non-zero integers  $r, s$ , both positive and negative. The braces  $\langle \dots \rangle$  appearing in (2.36) denote the average in  $E'$  over the interval  $[E - \Delta E, E + \Delta E]$ . Furthermore, the quantity  $l_H := 4\pi p \hbar \bar{d}$  is introduced. Apart from the factor of  $4\pi$ , which is subject to convention,  $l_H$  is a combination of quantities which has the dimension of a length and can be used as a ‘semiclassical parameter’ in that the semiclassical limit corresponds to  $l_H \rightarrow \infty$ . (The subscript refers to the fact that  $l_H$  is sometimes called ‘Heisenberg length’.) Notice that in the discussion following (1.63)  $l_H$  has been identified as the length where the periodic orbit sum (1.63) can be truncated without leaving out its most important contributions. Following this philosophy, we therefore also truncate the periodic orbit sums in (2.36) at  $|r l_n|, |s l_m| \leq l_H$ . Then obviously the resulting expression can at most represent the form factor in the range  $|\tau| \leq 1$ ; for larger values of  $|\tau|$  the truncated periodic orbit sum yields no contributions. But already in the interval  $\frac{1}{2} < \tau < 1$  some terms are missing so that a reliable statement can in fact only be made for  $|\tau| \leq \frac{1}{2}$ . Since now we are dealing with finite sums the semiclassical representation of  $K(\tau; E)$  is no longer plagued with convergence problems. The price to pay for this convenient fact is the restriction to  $|\tau| \leq \frac{1}{2}$ , and the inaccuracies which derive from the omission of the -infinite- tails of the periodic orbit sums. However, now the non-diagonal terms on the r.h.s. of (2.36), with  $r l_n \neq s l_m$ , are suppressed by the average  $\langle \dots \rangle$  over  $E'$ , or equivalently over  $p'$ . In the limit  $E, \Delta E \rightarrow \infty$  the non-diagonal terms actually vanish. We have therefore arrived at the semiclassical representation

$$K(\tau; E) \sim K_{diag}(\tau; E) := \frac{4p^2}{l_H^2} \sum_{l_n} \sum_{\substack{r \geq 1 \\ r l_n \leq l_H}} A_{n,r}^2 \delta \left( \tau - \frac{r l_n}{l_H} \right) \quad (2.38)$$

for  $0 \leq \tau \leq \frac{1}{2}$  and  $E \rightarrow \infty$ . The subscript refers to the fact that  $K_{diag}(\tau; E)$  is solely defined by the diagonal terms of the periodic orbit sums. Due to the exponential proliferation of the number of periodic orbits  $\gamma$  with lengths  $l_\gamma \leq l$ , see (1.64), the sum over  $r \geq 1$  is dominated by its contribution from  $r = 1$ . If we moreover substitute for the amplitude  $A_{n,1}$  its leading asymptotic behaviour (2.37) we obtain

$$K(\tau; E) \sim \frac{1}{l_H^2} \sum_{l_n \leq l_H} g_n^2 l_n^2 e^{-u_n} \delta \left( \tau - \frac{l_n}{l_H} \right). \quad (2.39)$$

The sum over the length spectrum in (2.39) will now be rewritten as a sum over periodic orbits, with the effect that one factor of  $g_n$  is being absorbed by the sum. We furthermore now assume that the multiplicities  $g_n$  of lengths  $l_n$  asymptotically approach a constant value of  $\bar{g}$ . Thus

$$K(\tau; E) \sim \frac{\bar{g}}{l_H^2} \sum_{\gamma, l_\gamma \leq l_H} l_\gamma^2 e^{-u_\gamma} \delta \left( \tau - \frac{l_\gamma}{l_H} \right). \quad (2.40)$$

We then integrate  $K(\tau'; E)$  with respect to  $\tau' \in [0, \tau]$ ,  $\tau \leq \frac{1}{2}$ ,

$$\int_0^\tau K(\tau', E) d\tau' \sim \frac{\bar{g}}{l_H^2} \sum_{\gamma, l_\gamma \leq l_H \tau} l_\gamma^2 e^{-u_\gamma}. \quad (2.41)$$

Periodic orbit sums like the one appearing in (2.41) frequently occur in the thermodynamic formalism for hyperbolic dynamical systems, see for example [47]. The asymptotic behaviour as  $l_H \rightarrow \infty$  of the r.h.s. of (2.41) can be obtained from relations known in the thermodynamic formalism, which relate the various dynamical entropies and exploit properties of the topological pressure  $P(\beta)$ , see (1.51). For hyperbolic systems with compact phase space one can in particular show that  $P(1) = 0$ , and from this result one can conclude that (2.41) yields  $\frac{\bar{g}}{2}\tau^2$ , so that after differentiation with respect to  $\tau$ ,

$$K(\tau; E) \sim \bar{g}\tau, \quad \tau \leq \frac{1}{2}, \quad E \rightarrow \infty. \quad (2.42)$$

The linear behaviour of the form factor was first obtained by Berry [46]. He concluded the result (2.42) from (2.40) with the help of a sum rule due to Hannay and Ozorio De Almeida [48].

Above we derived the general behaviour (2.33) of the form factor as  $\tau \rightarrow \infty$ . Since it must approach one, the linear increase (2.42) cannot extend beyond  $\tau \approx \frac{1}{2}$ . We recall that the restriction to  $\tau \leq \frac{1}{2}$  resulted from the truncation of the periodic orbit sums at  $l_H$ . It now becomes obvious that for large  $\tau$  the r.h.s. of (2.36) has to behave in a way that cannot be controlled by as simple semiclassical considerations as for small  $\tau$ . A further observation that can be made with the relation (2.42) is that the behaviour of  $K(\tau; E)$  depends on the average asymptotic multiplicities of lengths of periodic orbits. In ‘generic’ classical dynamical systems with time-reversal invariance one obtains that  $\bar{g} = 2$  because a periodic orbit and its time-reversed image share the same length. Here ‘generic’ means, among other things, that the classical dynamics have been completely desymmetrized, i.e., all discrete symmetries have been removed; a multiplicity of  $g_n \geq 3$  is then of an accidental kind. And, moreover, only if an orbit of length  $l_n$  is self-retracing, one observes  $g_n = 1$ . Typically, the asymptotic fraction of self-retracing orbits vanishes so that indeed ‘generically’  $\bar{g} = 2$ . For small  $l_n$ , however, effects of multiplicities  $g_n = 1$  remain. Hence, the relation (2.42) can only apply once one is deep enough in the semiclassical regime, i.e., for sufficiently large  $l_H$ . When the classical dynamics shows no time-reversal invariance, one typically finds  $g_n = 1$  so that then  $\bar{g} = 1$ . Thus quantum systems with ‘generic’ time-reversal invariant classical limits will yield a number variance that differs from those quantum systems whose classical limits are not time-reversal invariant. In random matrix theory a lack of time-reversal invariance forces one to deal with complex hermitian matrices and unitary base changes. Then the measure replacing the GOE is the Gaussian unitary ensemble (GUE).

At this point we can also hint at the reason for the exceptional behaviour of the spectral statistics in arithmetic quantum chaos, see [33, 34, 35, 36]. There the classical systems are completely chaotic, but the multiplicities of lengths of periodic orbits are exceptionally large; indeed one can show that  $g_n \sim c \frac{e^{l_n/2}}{l_n}$  for  $l_n \rightarrow \infty$ , [49, 50]. Clearly, the above discussion of the form factor has to be modified in an essential way. As a result, one can show [29] that  $K(\tau; E)$  increases exponentially instead of linearly.

We are now going to analyse the properties of the number variance by means of the relation (2.31). The form factor will be modelled by a simplified version that, however, captures its main features as they have been worked out above. For small  $\tau$ , close to zero, the form factor will be represented by its diagonal approximation (2.38), which we deliberately will sometimes use in the form (2.39) or (2.40). For larger values of  $\tau$ , however still below  $\frac{1}{2}$ ,  $K(\tau; E)$  can be well approximated by its linear overall increase (2.42) with  $\bar{g} = 2$ . Then, at  $\tau = \frac{1}{2}$ , both the approximations employed to arrive at (2.42) break down and the form factor reaches its asymptotic value of one, see (2.33). For simplicity, and because no further information on the behaviour of the form factor beyond  $\tau = \frac{1}{2}$  is available, we merely set  $K(\tau; E) = 1$  in our model. This therefore reads in explicit terms

$$K_M(\tau; E) := \begin{cases} K_{diag}(\tau; E) & , \quad 0 \leq \tau \leq \tau^* \\ 2\tau & , \quad \tau^* < \tau \leq \frac{1}{2} \\ 1 & , \quad \tau \geq \frac{1}{2} \end{cases} \quad (2.43)$$

see also [46]. At this point we remark that recently Bogomolny and Keating [51] managed to take non-diagonal contributions to (2.36) into account and hence provided a further justification for (2.43).

The value of  $\tau^*$ , beyond which we replace  $K_{diag}(\tau; E)$  by its semiclassical asymptotics (2.42), seems to be somewhat arbitrary. Its actual role will become clearer in the course of our subsequent discussion.

As a first illustration of the meaning of  $\tau^*$  we remark that (2.42) results from the estimate

$$\sum_{\gamma, l_\gamma \leq l} l_\gamma^2 e^{-u_\gamma} \sim \frac{1}{2} l^2, \quad l \rightarrow \infty, \quad (2.44)$$

applied to (2.41) with  $l = l_H \tau$ . As long as the cut-off  $l$  is not connected to  $l_H$  no quantum mechanical or semiclassical quantities appear; (2.44) thus is a purely classical asymptotics. One can hence introduce a classical length scale  $l_{cl}$  such that the r.h.s. of (2.44) approximates the l.h.s. to a given precision. We then define  $\tau^* := l_{cl}/l_H$ . Thus, for  $\tau \geq \tau^*$  one obtains that  $l_H \tau \geq l_{cl}$  which allows to use the asymptotics (2.44). The arbitrariness in the choice of  $\tau^*$  in (2.43) now appears as the -restricted-freedom to fix a value for  $l_{cl}$ . Certainly,  $l_{cl}$  has to be considerably larger than the length  $l_1$  of the shortest periodic orbit. Moreover, since  $\tau^* < \frac{1}{2}$  and indeed the semiclassical limit corresponds to  $l_H \rightarrow \infty$ , we have three well separated length scales  $l_1 \ll l_{cl} \ll l_H$ .

The model number variance now follows from (2.43) through

$$\Sigma_M^2(L; E) := \frac{2}{\pi^2} \int_0^\infty \frac{\sin^2(\pi L \tau)}{\tau^2} K_M(\tau; E) d\tau, \quad (2.45)$$

and consists of three parts which are due to the three  $\tau$ -ranges as they appear in (2.43). The first part derives from the integration over the interval  $0 \leq \tau \leq \tau^*$  and reads

$$\Sigma_{M,1}^2(L; E) \sim \frac{2}{\pi^2} \sum_{l_n} \sum_{\substack{k \geq 1 \\ kl_n \leq l_{cl}}} \frac{g_n^2}{k^2} e^{-ku_n} \sin^2\left(\frac{kl_n}{l_H} \pi L\right). \quad (2.46)$$

It depends on  $\tau^*$  through the cut-off  $l_{cl}$ . The second part, which emerges from the integration over  $\tau^* \leq \tau \leq \frac{1}{2}$ , is given by

$$\Sigma_{M,2}^2(L; E) = \frac{2}{\pi^2} [-\log(2\tau^*) - \text{Ci}(\pi L) + \text{Ci}(2\pi L \tau^*)], \quad (2.47)$$

recall (2.23) for the definition of  $\text{Ci}(x)$ . Finally, the remaining part reads

$$\Sigma_{M,3}^2(L; E) = \frac{2}{\pi^2} \left[ 1 - \cos(\pi L) - \pi L \text{Si}(\pi L) + \frac{\pi^2 L}{2} \right]. \quad (2.48)$$

An immediate observation that can be made with (2.46)–(2.48) is that  $\Sigma_M^2(L; E) \sim L$  for  $L \rightarrow 0$ , which is in accordance with the general statement (2.14).

Having explicit expressions at hand for the number variance, we are now in a position to analyse its behaviour for large  $L$ . To this end we first notice that  $\Sigma_M^2(L; E)$  can be split into a sum of two contributions, one of which is explicitly independent of  $\tau^*$  or  $l_{cl}$ , respectively, and reads

$$\begin{aligned} \Sigma_{M,\alpha}^2(L; E) &:= \frac{2}{\pi^2} \left[ 1 - \cos(\pi L) - \pi L \text{Si}(\pi L) + \frac{\pi^2 L}{2} + \log(2\pi L) - \text{Ci}(\pi L) \right] \\ &= \frac{2}{\pi^2} \log(2\pi L) + \frac{2}{\pi^2} + O\left(\frac{1}{L}\right), \quad L \rightarrow \infty. \end{aligned} \quad (2.49)$$

The second part

$$\Sigma_{M,\beta}^2(L; E) := \frac{2}{\pi^2} \left\{ \left[ \sum_{l_n} \sum_{\substack{k \geq 1 \\ kl_n \leq l_{cl}}} \frac{g_n^2}{k^2} e^{-ku_n} \sin^2\left(\frac{kl_n}{l_H} \pi L\right) \right] - \log(4\pi L \tau^*) + \text{Ci}(2\pi L \tau^*) \right\} \quad (2.50)$$

contains  $\tau^*$  and  $l_{cl}$ . We will see below, however, that the leading order asymptotics as  $L \rightarrow \infty$  is independent of the parameter  $l_{cl}$ . Inspecting (2.50) one observes that  $L$  and  $\tau^*$  appear in the combination  $L\tau^*$ . Now,  $\tau^* = l_{cl}/l_H \rightarrow 0$  in the semiclassical limit  $l_H \rightarrow \infty$ , which forces us to analyse the limit  $L \rightarrow \infty$  in the two cases  $L\tau^* \rightarrow 0$  and  $L\tau^* \rightarrow \infty$ . For convenience we now introduce the dimensionless quantity  $L_{max} := \frac{1}{\tau^*}$  so that in the following we discuss (i)  $L/L_{max} \rightarrow 0$  and (ii)  $L/L_{max} \rightarrow \infty$ .

In the first case the argument of  $\sin^2$  can be estimated as  $\frac{kl_n}{l_H}\pi L \leq \pi \frac{L}{L_{max}} \ll 1$  so that the periodic orbit sum in (2.50) yields a contribution of  $O((L/L_{max})^2)$ . Altogether, one obtains for (2.49) and (2.50) in the limit  $L \rightarrow \infty$ ,  $l_H \rightarrow \infty$ ,  $L/L_{max} \rightarrow 0$ ,

$$\Sigma_M^2(L; E) = \frac{2}{\pi^2} [\log(2\pi L) + 1 + \gamma - \log 2] + O\left(\left(\frac{L}{L_{max}}\right)^2\right) + O\left(\frac{1}{L}\right). \quad (2.51)$$

Our first observation with (2.51) is that, as announced before, all terms that have been evaluated explicitly do not depend on  $l_{cl}$ . Furthermore, a comparison with the number variance (2.22) in the GOE reveals that the leading terms coincide. Concerning the next-to-leading order, however, the constant in (2.51) contains  $-\log 2 \approx -0.693$  instead of  $-\frac{\pi^2}{8} \approx -1.234$  as in the GOE. Thus, our model (2.43) reproduces the number variance obtained in random matrix theory to leading order. The next-to-leading terms, however, differ slightly. This effect is most probably due to the crude approximation (2.43) to the actual form factor in the domain  $\tau \geq \frac{1}{2}$ . One might expect that the true form factor will fully reproduce the GOE result.

Remarkable deviations from random matrix theory show up in the second case  $L/L_{max} \rightarrow \infty$ . Here the arguments of the  $\sin^2$ -terms are no longer necessarily small; hence the periodic orbit sum contributes oscillatory terms. We are interested in the overall behaviour of the number variance as  $L \rightarrow \infty$ ,  $l_H \rightarrow \infty$ ,  $L/L_{max} \rightarrow \infty$  so that we now determine the mean value  $\Sigma_\infty^2(E)$  about which the oscillations of the number variance take place. This can be achieved by replacing each  $\sin^2$ -term by its average value  $\frac{1}{2}$ . Thus the periodic orbit sum no longer contains  $L$ ; it merely yields a constant  $C(l_{cl})$  that is solely determined by the value of the cut-off length  $l_{cl}$ . Collecting then the asymptotics of (2.49) and (2.50) in the limit  $L \rightarrow \infty$ ,  $l_H \rightarrow \infty$ ,  $L/L_{max} \rightarrow \infty$  yields

$$\Sigma_\infty^2(E) = \frac{2}{\pi^2} [\log L_{max} + 1 - \log 2 + C(l_{cl})] + O\left(\frac{L_{max}}{L}\right). \quad (2.52)$$

It now seems that the result (2.52) depends on  $l_{cl}$ . However, the following estimate shows that to leading order this dependence disappears. To this end take only the leading contribution of the primitive periodic orbits into account,

$$C(l_{cl}) \approx \frac{1}{\pi^2} \sum_{l_n \leq l_{cl}} g_n^2 e^{-u_n} \approx \frac{\bar{g}}{\pi^2} \sum_{\gamma, l_\gamma \leq l_{cl}} e^{-u_\gamma}. \quad (2.53)$$

Then, in analogy to (2.44) the thermodynamic formalism allows to estimate the r.h.s. of (2.53) for large  $l_{cl}$ . Together with  $\bar{g} = 2$  one thus observes that  $C(l_{cl}) \sim \frac{2}{\pi^2} \log l_{cl}$ ,  $l_{cl} \rightarrow \infty$ , with the effect that finally

$$\Sigma_\infty^2(E) \approx \frac{2}{\pi^2} [\log l_H + 1 - \log 2], \quad (2.54)$$

where now  $l_{cl}$  no longer appears. However, if one wants to calculate a numerical value for  $\Sigma_\infty^2(E)$  the approximations employed might not be satisfactory so that the value of the constant on the r.h.s. of (2.52) cannot really be fixed by our considerations. The relation (2.52) then only shows that this constant depends on the distribution of the short periodic orbits (with lengths  $\leq l_{cl}$ ). If in addition we introduce the explicit form  $l_H = 4\pi\hbar\bar{d}\sqrt{E}$  for the Heisenberg length, we can obtain from (2.52) the following high energy asymptotics,

$$\Sigma_\infty^2(E) \sim \frac{1}{\pi^2} \log E, \quad E \rightarrow \infty. \quad (2.55)$$

If we now compare (2.51) and (2.52) we realize that the number variance, as a function of  $L$ , grows logarithmically up to approximately  $L \approx L_{max}$ . Beyond this scale  $\Sigma^2(L; E)$  ceases to increase, but rather oscillations caused by classical periodic orbits set in which take place about the saturation value  $\Sigma_\infty^2(E)$ . Both the transition scale  $L_{max} = l_H/l_{cl}$  and the saturation value depend on  $E$ : The larger  $E$ , the larger are both  $L_{max}$  and  $\Sigma_\infty^2(E)$ . Since the number variance is ‘generically’ well described by random matrix theory in the domain  $L \ll L_{max}$ , this is sometimes called the universality regime. On the other hand, the oscillations of the number variance about  $\Sigma_\infty^2(E)$  in the domain  $L \gg L_{max}$  suggest to call this the saturation regime. This general picture has been confirmed numerically in the case of the zeros of the Riemann zeta function [30] and for some two dimensional classically chaotic systems, e.g. in [31, 52].

As a final remark on the number variance let us compare the results of the semiclassical analysis with the general observations made in section 2.1. To this end we first have to recall the connection between the variables  $E$  and  $x$ : since  $x = \overline{N}(E) \sim \frac{A}{4\pi\hbar^2} E = \overline{d}E$ , one observes that  $l_H = 4\pi\hbar\overline{d}\sqrt{E} \sim 4\pi\hbar\sqrt{\overline{d}x} = \sqrt{4\pi A}\sqrt{x}$ . While discussing the relation (2.15) we noticed that the limit  $L \rightarrow \infty$  has to be considered in conjunction with  $x \rightarrow \infty$ . Due to the above the latter is equivalent to the semiclassical limit  $l_H \rightarrow \infty$ . We also related the two limits in choosing  $L = ax^\gamma$  with some exponent  $0 < \gamma < 1$ . Since  $L_{max} = l_H/l_{cl} = \text{const.}\sqrt{x}$ , the two cases  $L/L_{max} \rightarrow 0$  and  $L/L_{max} \rightarrow \infty$ , which we were forced to distinguish in the semiclassical analysis, therefore correspond to  $0 < \gamma < \frac{1}{2}$  and  $\frac{1}{2} < \gamma < 1$ , respectively. We hence now have a means to characterize what we before alluded to as a large exponent  $\gamma$ . The latter should result in a decoupling of the two random variables  $N_{fl}(y)$  and  $N_{fl}(y + L)$  on  $I_x$ , when  $x \rightarrow \infty$  and  $L = ax^\gamma$ . Now, choosing  $\gamma$  larger than the critical exponent  $\gamma_c = \frac{1}{2}$  corresponds to  $L/L_{max} \rightarrow \infty$  and therefore to the saturation regime of the number variance. Here  $\Sigma^2(L; E)$  oscillates about the  $L$ -independent saturation value  $\Sigma_\infty^2(E)$ . Recalling the relation (2.15), we hence conclude that the oscillations can only be caused by the correlation function on the r.h.s. of (2.15), whereas the  $L$ -independent contribution  $\Sigma_\infty^2(E)$  comes from the first two terms on the r.h.s. Moreover, (2.55) implies that

$$\mathbb{E}_x [N_{fl}^2] \sim \frac{1}{2\pi^2} \log x, \quad x \rightarrow \infty, \quad (2.56)$$

compare also (2.16). So far the whole analysis concerned two dimensional systems. An immediate generalization to the  $d$  dimensional case reveals that  $L_{max} \sim \text{const.} x^{1-\frac{1}{d}}$ , so that the critical exponent  $\gamma_c$  above which saturation takes place is in general  $\gamma_c = 1 - \frac{1}{d}$ . Thus, in larger dimension the saturation regime for the limit  $L \rightarrow \infty$  becomes smaller. However, it could only disappear in infinite dimensional systems.

The above analysis therefore leads us to conjecture that the two random variables  $N_{fl}(y)$  and  $N_{fl}(y + L)$  on  $I_x$  become statistically independent in the limit  $x \rightarrow \infty$  when  $L = ax^\gamma$  with  $\gamma > \gamma_c$ . Once  $\gamma < \gamma_c$ , strong correlations remain that force the number variance to follow the predictions of the GOE. For certain two dimensional classically integrable systems Bleher and Lebowitz [53] proved that indeed the saturation behaviour, with  $\mathbb{E}_x[N_{fl}^2] \sim \text{const.}\sqrt{x}$ , occurs as well as the statistical independence for  $\gamma > \gamma_c = \frac{1}{2}$ . The difference in  $\mathbb{E}_x[N_{fl}^2]$  between classically integrable and chaotic systems (2.56) stems from the occurrence of periodic orbits as one-parameter families in the former case. Under certain regularity assumptions, Sarnak [36] showed that the existence of one-parameter families of closed geodesics in the unit tangent bundle over a surface implies  $\Sigma^2(L; E) \gg \sqrt{E}$  for  $L \sim E \rightarrow \infty$ . To this end he used the trace formula in order to represent  $N_{fl}(E)$ , and in particular the structure (1.35) implied by  $k$ -parameter families which yield exponents  $m_j = k + 1$ . In the example of the stadium billiard, where the bouncing ball orbits occur as a one-parameter family that can be dealt with in full detail, Sieber et al. [54] determined the saturation value of the number variance as  $\Sigma_\infty^2(E) = \text{const.}\sqrt{E}$ , and thus confirmed the effect of non-isolated periodic orbits even in classically chaotic systems.



### 3 From Local to Global Eigenvalue Statistics

The principal measure for the distribution of the eigenvalues of some quantum Hamiltonian that was discussed in section 2 was the probability  $E(k; L)$  to observe  $k$  defolded eigenvalues in an interval of length  $L$ . As long as  $L$  is finite one thus measures correlations on a scale  $L$ , i.e., among approximately  $L$  eigenvalues. One is therefore dealing with local eigenvalue statistics. If one wants to include correlations among infinitely many levels, one is forced to consider the limit  $L \rightarrow \infty$ . At several stages in the previous discussion we already considered asymptotics for large  $L$ , and hence the transition to the global scale. It is the goal of the present section to investigate this limit in some more detail. We will in particular emphasize the distinction as to whether one passes to the global scale within the universality regime  $L/L_{max} \rightarrow 0$ , or in the saturation regime  $L/L_{max} \rightarrow \infty$ . It will then be noticed that the latter approach offers an interesting opportunity to identify fingerprints of classical chaos in the distribution of eigenvalues.

#### 3.1 The Renormalized Random Variable

Our discussion of the local eigenvalue statistics in section 2 was based on the random variable  $n_L(y) = N(y + L) - N(y)$  for  $y \in I_x = [x - \Delta x, x + \Delta x]$ . We noticed that its expectation value reads  $\mathbb{E}_x[n_L] \sim L$ ,  $x \rightarrow \infty$ , see (2.9). Moreover, its variance is essentially given by the number variance (2.12), which grows logarithmically for classically chaotic and linearly for classically integrable systems as  $L \rightarrow \infty$  once one confines oneself to the universality regime  $L \ll L_{max} = \text{const.} \sqrt{x}$ , see (2.51). In the saturation regime  $L \gg L_{max}$  it oscillates about  $\Sigma_\infty^2$ . In any case, both the mean value and the variance of the random variable  $n_L(y)$  diverge as  $L \rightarrow \infty$ ,  $x \rightarrow \infty$ . One is therefore advised to renormalize the random variable so as to yield finite first and second moments in order to have a chance to obtain a finite limit distribution. The most simple choice

$$\eta_L(y) := \frac{n_L(y) - L}{\sqrt{\Sigma_x^2(L)}} , \quad y \in I_x , \quad (3.1)$$

certainly ensures that the lowest two moments are finite, since

$$\mathbb{E}_x[\eta_L] = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \eta_L(y) dy = \frac{\mathbb{E}_x[n_L] - L}{\sqrt{\Sigma_x^2(L)}} \rightarrow 0 , \quad (3.2)$$

for  $x \rightarrow \infty$ , and

$$\mathbb{E}_x[\eta_L^2] = \frac{\mathbb{E}_x[(n_L - L)^2]}{\Sigma_x^2(L)} = 1 . \quad (3.3)$$

In general, the  $m$ -th moments of the renormalized random variable  $\eta_L(y)$  can be expressed in terms of the moments  $\Sigma_x^m(L)$ , see (2.11), of  $n_L(y) - L$ ,

$$M_x^m(L) := \mathbb{E}_x[\eta_L^m] = \frac{1}{[\Sigma_x^2(L)]^{\frac{m}{2}}} \mathbb{E}_x[(n_L - L)^m] = \frac{\Sigma_x^m(L)}{[\Sigma_x^2(L)]^{\frac{m}{2}}} . \quad (3.4)$$

Due to the normalization of  $\eta_L(y)$ , as expressed by (3.2) and (3.3), one is hence led to expect a limit distribution for  $L \rightarrow \infty$  to exist.

The characteristic function of  $\eta_L(y)$  can now be expressed in terms of the characteristic function of  $n_L(y)$ ,

$$J_{\eta_L, x}(\xi) = \mathbb{E}_x[e^{i\xi\eta_L}] = e^{-i\xi \frac{L}{\sqrt{\Sigma_x^2(L)}}} J_{n_L, x}\left(\frac{\xi}{\sqrt{\Sigma_x^2(L)}}\right) , \quad (3.5)$$

compare (2.17). This quantity allows to characterize the distribution of the random variable  $\eta_L(y)$  through

$$\begin{aligned}
\nu_{\eta_L, x}(d\eta) &= \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \delta(\eta_L(y) - \eta) dy d\eta \\
&= \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i[\eta_L(y)-\eta]\xi} d\xi dy d\eta \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\eta\xi} J_{\eta_L, x}(\xi) d\xi d\eta .
\end{aligned} \tag{3.6}$$

Going back to (2.18), we realize that

$$\begin{aligned}
\sqrt{\Sigma_x^2(L)} E_x \left( \eta \sqrt{\Sigma_x^2(L)} + L; L \right) d\eta &= \frac{\sqrt{\Sigma_x^2(L)}}{2\pi} \int_{-\pi}^{+\pi} e^{-i\rho(\eta\sqrt{\Sigma_x^2(L)}+L)} J_{n_L, x}(\rho) d\rho d\eta \\
&= \frac{1}{2\pi} \int_{-\pi\sqrt{\Sigma_x^2(L)}}^{+\pi\sqrt{\Sigma_x^2(L)}} e^{-i\xi\eta} J_{\eta_L, x}(\xi) d\xi d\eta ,
\end{aligned} \tag{3.7}$$

where we changed variables from  $\rho$  to  $\xi := \rho\sqrt{\Sigma_x^2(L)}$  and employed (3.5). Hence a comparison of (3.6) and (3.7) yields

$$\nu_{\eta_L, x}(d\eta) \sim \sqrt{\Sigma_x^2(L)} E_x \left( \eta \sqrt{\Sigma_x^2(L)} + L; L \right) d\eta , \tag{3.8}$$

whenever the number variance approaches infinity, i.e., when  $L \rightarrow \infty$ ,  $x \rightarrow \infty$ . Since we concluded (3.8) from the relation (3.5) of the characteristic functions, the asymptotic equivalence of both sides of (3.8) is to be understood in the sense of weak limits. This means that for any bounded continuous function  $g(\eta)$

$$\lim_{\substack{x \rightarrow \infty \\ L \rightarrow \infty}} \int_{-\infty}^{+\infty} g(\eta) \nu_{\eta_L, x}(d\eta) = \lim_{\substack{x \rightarrow \infty \\ L \rightarrow \infty}} \sqrt{\Sigma_x^2(L)} \int_{-\infty}^{+\infty} g(\eta) E_x \left( \eta \sqrt{\Sigma_x^2(L)} + L; L \right) d\eta . \tag{3.9}$$

As a first example, let us discuss a Poissonian random process  $n_L(y)$ . Since this is stationary there appears no dependence of its distribution on  $x$ . The probability to find  $k$  events (i.e., eigenvalues) in an interval of length  $L$  is then given by

$$E_{Poisson}(k; L) = \frac{L^k}{k!} e^{-L} . \tag{3.10}$$

Now, it is well known and easy to see that (3.10) approaches a Gaussian for  $L \rightarrow \infty$ ,

$$E_{Poisson}(k; L) \sim \frac{1}{\sqrt{2\pi L}} e^{-\frac{(k-L)^2}{2L}} = \frac{1}{\sqrt{2\pi \Sigma^2(L)}} e^{-\frac{(k - \mathbb{E}[n_L])^2}{2\Sigma^2(L)}} , \tag{3.11}$$

since  $\mathbb{E}_{Poisson}[n_L] = L$  and  $\Sigma_{Poisson}^2(L) = L$ . Hence, according to (3.8) the distribution of  $\eta_L(y)$  approaches

$$\nu_{\eta_L}(d\eta) \rightarrow \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\eta^2} d\eta , \quad L \rightarrow \infty . \tag{3.12}$$

In random matrix theory one obtains a similar asymptotic result. Here the random process  $n_L(y)$  is also stationary, after one has performed the limit of matrix dimension  $N \rightarrow \infty$ , and hence no  $x$ -dependence occurs either. In section 2.2 we already reported on the numerical finding (2.24). Due to the relation (3.8) one therefore would conclude that (3.12) also holds for the GOE. Indeed, Costin and Lebowitz [39] calculated the cumulants of the distribution of  $\eta_L(y)$  in the GOE, GUE, and GSE (Gaussian symplectic ensemble) and proved that  $\nu_{\eta_L}(d\eta)$  weakly converges to a Gaussian with zero mean and unit variance.

At this point we recall that the stationarity of the random process in the Poissonian as well as in the random matrix case makes the limit  $L \rightarrow \infty$  unique, i.e., no saturation effects and related phenomena occur. As we saw in section 2.3 the situation is different for spectra of individual quantum Hamiltonians. Now, provided one accepts that the local eigenvalue distributions of ‘generic’ quantum Hamiltonians are described by Poisson statistics in the classically integrable case and by random matrix theory in the chaotic case, and that this description extends to hold when passing to the global scale within the universality regime, the above findings suggest that

$$\lim_{L \rightarrow \infty} \int_{-\infty}^{+\infty} g(\eta) \nu_{\eta_L, x}(d\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\eta) e^{-\frac{1}{2}\eta^2} d\eta , \quad (3.13)$$

for  $L \rightarrow \infty$ ,  $x \rightarrow \infty$ ,  $L/L_{max} \rightarrow 0$ , and any bounded continuous function  $g(\eta)$ . Thus, a central limit theorem would hold true for the global distribution of eigenvalues in the universality regime, irrespective of the type of classical dynamics.

As can be anticipated from the discussion in section 2.3, the situation changes drastically, if one passes to the global scale within the saturation regime  $L/L_{max} \rightarrow \infty$ . In 2.3 we interpreted the saturation phenomenon in terms of the relation (2.15) for the number variance and concluded that all evidence seems to be in favour of an asymptotic statistical independence of the two random variables  $N_{fl}(y+L)$  and  $N_{fl}(y)$ ,  $y \in I_x$ , as  $L/L_{max} \rightarrow \infty$ . We remark that instead of choosing the arguments  $y+L$  and  $y$ , with  $y \in I_x$ , one can obviously also consider  $N_{fl}(y)$  with  $y \in I_x$  and  $y \in I_{x+L}$ , respectively. In analogy to the discussion above, one should also renormalize these random variables and hence introduce

$$W(y) := \frac{N_{fl}(y)}{\sqrt{\mathbb{E}_x[N_{fl}^2]}} , \quad (3.14)$$

for  $y \in I_x$  and  $y \in I_{x+L}$ , respectively. The first and second moments of this random variable are clearly analogous to (3.2) and (3.3),

$$\mathbb{E}_x[W] \rightarrow 0 , \quad x \rightarrow \infty , \quad \text{and} \quad \mathbb{E}_x[W^2] = 1 . \quad (3.15)$$

In view of the saturation of the number variance,

$$\Sigma_x^2(L) \sim \Sigma_\infty^2 \sim 2 \mathbb{E}_x[N_{fl}^2] \sim 2 \mathbb{E}_{x+L}[N_{fl}^2] \quad (3.16)$$

for  $x \rightarrow \infty$ ,  $L \rightarrow \infty$ ,  $L/L_{max} \rightarrow \infty$ , the random variable

$$\eta_L(y) \sim \frac{N_{fl}(y+L) - N_{fl}(y)}{\sqrt{\Sigma_\infty^2}} \quad (3.17)$$

appears as the difference of the asymptotically independent random variables  $W(y)$ ,  $y \in I_x$ , and  $W(y)$ ,  $y \in I_{x+L}$ . Thus, instead of the limit distribution of  $\eta_L(y)$  as  $L/L_{max} \rightarrow \infty$ , one can equivalently consider the limit distribution of  $W(y)$  as  $x \rightarrow \infty$ . Provided the statistical independence alluded to above indeed holds, both limit distributions coincide.

The quantity  $W(y)$  has a nice and simple interpretation in that it describes the -normalized- fluctuations of the spectral staircase  $N(E)$  about its mean behaviour  $\overline{N}(E)$ . Furthermore, it can easily be calculated from numerical data, and its distribution is readily computed. It thus appears that the global eigenvalue distribution in the saturation regime is most conveniently measured in terms of the limit distribution of  $W(y)$  for large  $x$ .

## 3.2 Classically Integrable Systems

In contrast to the case of classically chaotic systems, spectra of quantum Hamiltonians that arise as quantizations of integrable classical systems allow for a fairly explicit treatment. Due to the

semiclassical EBK-quantization scheme the behaviour of the spectral staircase  $N(E)$  can be related to a lattice point problem. Given an integrable classical system with  $d$  degrees of freedom, one introduces action-angle variables  $(I, \theta)$ . The classical Hamiltonian function then depends only on the actions,  $H(I) = H(I_1, \dots, I_d)$ . According to the EBK-procedure, semiclassical approximations to the quantum energies can be obtained once one quantizes the actions as  $I_k = (n_k + \frac{\alpha_k}{4})\hbar$ . Here  $n_k \in \mathbb{Z}$ , or  $n_k \in \mathbb{N}$ , yields a quantum number, and  $\alpha_k \in \{0, 1, 2, 3\}$  is an appropriate Maslov index. Then

$$E_{n_1, \dots, n_d}^{EBK} := H\left(\left(n_1 + \frac{\alpha_1}{4}\right)\hbar, \dots, \left(n_d + \frac{\alpha_d}{4}\right)\hbar\right) \quad (3.18)$$

defines semiclassical energies which approximate the eigenvalues of the quantum Hamiltonian  $\hat{H}$  for small  $\hbar$ . The EBK-spectral staircase

$$N^{EBK}(E) := \#\{(n_1, \dots, n_d); E_{n_1, \dots, n_d}^{EBK} \leq E\} \quad (3.19)$$

counts the number of lattice points  $(n_1, \dots, n_d)$  inside the domain that is bordered by the  $(d-1)$ -dimensional manifold defined by  $H(I_1, \dots, I_d) = E$ . As usual in lattice point problems, the leading behaviour of  $N^{EBK}(E)$  in the limit  $E \rightarrow \infty$  is provided by the volume of the manifold which is given by the condition  $H(I_1, \dots, I_d) \leq E$ . If one then takes the complete phase space into account by introducing an additional integration over the angle variables one observes that

$$N^{EBK}(E) \sim \frac{1}{(2\pi\hbar)^d} \int \int \Theta(E - H(I)) \, dI \, d\theta. \quad (3.20)$$

The r.h.s. is identical to the r.h.s. of (1.2) because the transformation  $(I, \theta) \mapsto (p, x)$  is canonical and hence preserves phase space volumes. Therefore (3.19) yields the actual spectral staircase at least to leading order as  $\hbar \rightarrow 0$ . However, as discussed at length in section 2 the fine structure in the eigenvalue distribution, which is of interest in quantum chaos, is encoded in the remainder term to the leading asymptotics (1.2) of the spectral staircase. In order to analyse spectral statistics in terms of the lattice point problem (3.19) one therefore has to ensure that the limit distributions arising from  $N(E)$  and  $N^{EBK}(E)$ , respectively, coincide. For the free motion of a particle on a surface of revolution that obeys a certain non-degeneracy condition Bleher [55] proved this to be the case. The same fact was proven by Kosygin et al. [56] for the free motion on a torus with Liouville metric, which is also integrable.

As a first and very simple example of a classically integrable system let us discuss a one dimensional billiard, i.e., the free motion on the interval  $[0, l]$  with elastic reflections from the boundary points at 0 and  $l$ . In units where  $\hbar = 1 = 2m$  the eigenvalues of the quantum Hamiltonian  $\hat{H} = -\frac{d^2}{dx^2}$  with Dirichlet boundary conditions read  $E_n = \frac{\pi^2}{l^2}n^2$ ,  $n \in \mathbb{N}$ . If we denote the integer part of a real number  $r$  by  $[r]$ , the spectral staircase is given by  $N(E) = [\frac{l}{\pi}\sqrt{E}] \sim \frac{l}{\pi}\sqrt{E}$ ,  $E \rightarrow \infty$ . As the defolded energy variable we thus introduce  $x := \frac{l}{\pi}\sqrt{E}$ . Then the fluctuations of the spectral staircase read

$$N_{fl}(x) = \frac{1}{2} - \{x\} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi nx), \quad (3.21)$$

where  $\{x\} = x - [x]$  denotes the fractional part of  $x$ . The  $\frac{1}{2}$  appearing in the middle term of (3.21) is chosen such that  $N_{fl}(x)$  fluctuates about zero. Clearly,  $|N_{fl}(x)| \leq \frac{1}{2}$ , and the r.h.s. of (3.21) constitutes a representation of the one-periodic and piecewise continuous function  $N_{fl}(x)$  by its Fourier series. The variance of  $N_{fl}(x)$  can easily be determined to yield  $\mathbb{E}_x[N_{fl}^2] \sim \frac{1}{12}$  for  $x \rightarrow \infty$ , so that  $W(y) = \sqrt{12}N_{fl}(y)$ . The characteristic function for the distribution of  $W(y)$  is given by

$$J_{W,x}(\xi) = \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} e^{i\xi W(y)} dy. \quad (3.22)$$

In the limit  $x \rightarrow \infty$  we obtain

$$J_W(\xi) = \int_0^1 e^{i\xi\sqrt{12}(\frac{1}{2}-y)} dy = \frac{1}{\sqrt{3}\xi} \sin(\sqrt{3}\xi) = \frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{+\sqrt{3}} e^{i\xi w} dw . \quad (3.23)$$

Since the density  $p(w)$  of the limit distribution of  $W(y)$  is related to the characteristic function by Fourier inversion, we observe from (3.23) that

$$p(w) = \begin{cases} \frac{1}{2\sqrt{3}} & , \quad -\sqrt{3} \leq w \leq +\sqrt{3} \\ 0 & , \quad |w| > \sqrt{3} \end{cases} . \quad (3.24)$$

Thus,  $W(y)$  is asymptotically uniformly distributed on a finite interval. The fact that  $p(w) = 0$  for  $|w| > \sqrt{3}$  results from the bound  $|W(y)| = \sqrt{12}|N_{fl}(y)| \leq \sqrt{3}$ ; this clearly determines the support of the probability density  $p(w)$ . The result (3.24) could also be naively anticipated from the explicit form and the periodicity of  $N_{fl}(x)$ .

A more prototypical example, however, of a classically integrable system is provided by the free motion of a particle on a two dimensional torus. As discussed at the end of section 2.2, the quantum energies for this system are exactly represented by an expression that is proportional to the positive definite binary quadratic form (2.26). The spectral staircase  $N(E)$  is hence exactly given by a lattice point problem; more precisely,  $N(E)$  is the number of points of the lattice  $\mathbb{Z}^2$  inside an ellipse that is defined by the quadratic form  $Q$ . In this example we again choose units such that  $\hbar = 1 = 2m$ .

Let us now discuss the torus  $\mathbb{R}^2/\mathcal{L}$  with  $\mathcal{L} = \mathbb{Z}^2$  in some more detail. Here  $N(E)$  is the number of lattice points inside a circle of radius  $\frac{\sqrt{E}}{2\pi}$ , i.e., we are dealing with the circle problem. Employing the Poisson summation formula, one can readily demonstrate that for  $E \rightarrow \infty$

$$\begin{aligned} N(E) &= \frac{E}{4\pi} + \frac{E^{\frac{1}{4}}}{\sqrt{2\pi^3}} \sum_{(n_1, n_2) \neq (0,0)} \frac{1}{(n_1^2 + n_2^2)^{\frac{3}{4}}} \cos\left(\sqrt{n_1^2 + n_2^2} \sqrt{E} - \frac{3\pi}{4}\right) + O\left(E^{-\frac{1}{4}}\right) , \\ &= \frac{E}{4\pi} + \frac{E^{\frac{1}{4}}}{\sqrt{2\pi^3}} \sum_{\substack{m \in \mathbb{N} \\ m \text{ square free}}} \frac{1}{m^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} \cos\left(k\sqrt{m} \sqrt{E} - \frac{3\pi}{4}\right) + O\left(E^{-\frac{1}{4}}\right) . \end{aligned} \quad (3.25)$$

Since  $\sum_{n_1, n_2} (n_1^2 + n_2^2)^{-\frac{3}{4}}$  diverges, one cannot estimate the sum on the r.h.s. of (3.25) in a simple way as a function of  $E$ ; to obtain improved bounds for this sum still constitutes an active area of research in analytic number theory. A novel perspective on the circle problem, which inspired much of the work on the global distribution of quantum energies, was introduced by Heath-Brown [57]. Instead of estimates of the remainder term in the circle problem, he considered its distribution. As we will soon realize, this corresponds to the limit distribution of  $W(y)$ . According to (3.25), where the first term on the r.h.s. defines  $\overline{N}(E)$ , the defolded spectral variable reads  $x = \frac{E}{4\pi}$ . Thus

$$N_{fl}(x) = \left(\frac{x}{\pi}\right)^{\frac{1}{4}} \Phi(x) + O\left(x^{-\frac{1}{4}}\right) , \quad x \rightarrow \infty , \quad (3.26)$$

with

$$\Phi(x) = \sum_{\substack{m \in \mathbb{N} \\ m \text{ square free}}} f_m\left(\sqrt{\frac{x}{\pi}} \sqrt{m}\right) , \quad (3.27)$$

where

$$f_m(t) := \frac{1}{\pi m^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} \cos\left(2\pi kt - \frac{3\pi}{4}\right) . \quad (3.28)$$

It is furthermore known, see for example [57], that  $\mathbb{E}_x[N_{fl}^2] \sim \text{const.} \sqrt{x}$ ,  $x \rightarrow \infty$ , so that  $W(y) = \text{const.} \Phi(y)$ . Apart from the normalization of the variance, the limit distributions of  $W(y)$  and  $\Phi(y)$

therefore coincide. Notice that the representation (3.25) of the spectral staircase can be interpreted in terms of a semiclassical trace formula. To this end one observes that  $\sqrt{n_1^2 + n_2^2}$ , with  $(n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ , yields all lengths of periodic orbits on the torus. In the same spirit, the sum in (3.27) can be identified as extending over all lengths  $\sqrt{m}$  of primitive periodic orbits; their repetitions are then accounted for by the sum over  $k$  in (3.28).

The functions  $f_m(t)$  in (3.28) are clearly continuous and periodic with period 1. For such functions Heath-Brown proved [57] that whenever  $\gamma_1, \dots, \gamma_k$  are real numbers linearly independent over  $\mathbb{Q}$ , then  $f_{m_1}(\gamma_1 t), \dots, f_{m_k}(\gamma_k t)$ , with  $0 \leq t \leq T$ , become statistically independent in the limit  $T \rightarrow \infty$ . If one now chooses  $\gamma_m = \sqrt{m}$ ,  $m$  square free, any set  $\{\gamma_{m_1} \neq \dots \neq \gamma_{m_k}\}$  is indeed linearly independent implying that the random variables  $f_m(t\sqrt{m})$ ,  $0 \leq t \leq T$ , as they appear in (3.27), asymptotically become independent for  $T \rightarrow \infty$ . A further analysis reveals [28, 58] that under fairly general assumptions about a set of continuous real valued periodic functions  $b_1(t), b_2(t), \dots$  with

$$\int_0^1 b_n(t) dt = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \int_0^1 b_n(t)^2 dt < \infty, \quad (3.29)$$

the series  $F(t) = \sum_n b_n(\gamma_n t)$ ,  $0 \leq t \leq T$ , has a distribution that weakly converges to the distribution of the random series  $\sum_n b_n(\theta_n)$ , where the  $\theta_n$ 's are independent random variables uniformly distributed on the interval  $[0, 1]$ . The latter fact arises because the periodicity of the functions  $b_n(t)$  results in  $\gamma_n t$  to be taken mod 1. As  $t$  varies over the interval  $[0, T]$ , the values of  $\gamma_n t$  mod 1 become uniformly distributed on  $[0, 1]$  as  $T \rightarrow \infty$ .

We remark that for the following results to hold the second condition in (3.29) becomes essential; it roughly means that the maximum of  $b_n(t)$  must decrease sufficiently fast as  $n \rightarrow \infty$ . Then a sufficient condition for the existence of the limit distribution is for example given by [57, 28]

$$\lim_{N \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \min \left\{ 1, \left| F(t) - \sum_{n \leq N} b_n(\gamma_n t) \right| \right\} dt = 0. \quad (3.30)$$

This means that  $F(t)$  can be represented, in the sense indicated in (3.30), by the infinite sum  $\sum_n b_n(\gamma_n t)$ . Under further assumptions on the behaviour of the functions  $b_n(t)$  it was moreover proven [28] that the limit distribution has a density  $p(x)$  which behaves asymptotically as  $e^{-c|x|^\rho}$  for  $x \rightarrow \pm\infty$ . For the circle problem it was verified [28] that the functions  $f_m(t)$  meet all requirements, and the exponent was estimated from above and below by  $\rho_{\pm} = 4 \pm \varepsilon$  for any  $\varepsilon > 0$ , respectively; thus the limit distribution is not Gaussian.

We are interested in the limit distribution of  $W(y)$  for the eigenvalues of the quantum Hamiltonian on the torus  $\mathbb{R}^2/\mathbb{Z}^2$ , and hence in the limit distribution of  $\Phi(y)$ , see (3.26). By the change of variables  $x = \pi t^2$  we can relate this problem to the distribution of the random series  $\sum_m f_m(\theta_m)$  and then apply the results of [57, 28]. Therefore

$$\lim_{x \rightarrow \infty} \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} g(W(y)) \rho\left(\frac{y-x}{2\Delta x} + \frac{1}{2}\right) dy = \int_{-\infty}^{+\infty} g(w) p(w) dw \quad (3.31)$$

for any piecewise continuous bounded function  $g(w)$  and any probability density  $\rho(\tau)$ ,  $\tau \in [0, 1]$ . Here  $p(w)$  is the non-Gaussian probability density mentioned above. The freedom to choose a weight  $\rho(\tau)$  allows to change variables from, say,  $y$  to  $t$  and thus to compensate for the Jacobian then arising.

The existence of a limit distribution (3.31) has by now been established for a large class of two dimensional classically integrable systems, see [59] for a review. This includes the free motion on more general flat tori, with and without Aharonov-Bohm type magnetic fluxes [28, 60], on surfaces of revolution [55], and on tori with Liouville metrics [56, 61]. All these cases lead to limit distributions with densities  $p(w)$  that approximately decay as  $e^{-c|w|^4}$  for  $w \rightarrow \pm\infty$ .

In a further class of examples for two dimensional classically integrable systems the limit distributions of  $W(y)$  were explicitly determined by Schubert [62]. He considered the geodesic motion on

Zoll surfaces, i.e., surfaces with the topology of a sphere which are endowed with a metric all of whose geodesics are closed, and all lengths of periodic orbits are multiples of a fundamental length. The simplest example for a Zoll surface is the sphere with its usual round metric; however, non-trivial deformations thereof exist. Schubert showed that  $N_{fl}(x) = \sqrt{x}\Theta(\sqrt{x})$ , where  $\Theta(t)$  is an almost periodic function of Besicovitch-class  $B^2$ , see [58] for explanations. The trace formula then allows to represent  $\Theta(t)$  by an  $L^2$ -convergent Fourier series that is, apart from a possibly non-trivial Maslov phase, identical to (3.21). Therefore,  $W(y) = \Theta(\sqrt{y})$  for Zoll surfaces has the same limit distribution (3.24) as a one dimensional billiard. This seemingly surprising coincidence is caused by the fact that in both cases the classical side of the trace formula is given by a sum over multiples of one fundamental length. The extra dimension of a Zoll surface thus is only reflected in the overall growth of  $N_{fl}(x)$  by  $\sqrt{x}$ . Notice also the difference to the case of tori where the overall growth of  $N_{fl}(x)$  is given by  $x^{\frac{1}{4}}$ .

As a final remark we observe that the decay of the propability density  $p(w)$  as  $w \rightarrow \pm\infty$  is connected to the type of convergence of the periodic orbit sum representing  $N_{fl}(x)$ . For the one dimensional billiard as well as for Zoll surfaces the series converges in  $L^2$ -norm, which is considerably strong. In contrast, one obtains only the weaker form of convergence (3.30) for flat tori. We hence conclude that the weaker the convergence is, the weaker the decay of  $p(w)$ . This observation will be important for classically chaotic systems. We also notice that in all examples of classically integrable systems studied so far the distribution of  $W(y)$  has a limit as  $x \rightarrow \infty$  that is non-Gaussian; thus in the integrable case no central limit theorem for the global distribution of eigenvalues in the saturation regime is valid.

### 3.3 Classically Chaotic Systems

In analogy to the situation for classically integrable systems that has been described above, the global distribution of eigenvalues of quantum Hamiltonians with chaotic classical limits will now be studied by means of the limit distributions of  $W(y)$ ,  $y \in I_x$ , as  $x \rightarrow \infty$ . Above we realized that these distributions can be obtained from periodic orbit sums for  $N_{fl}(x)$ . In all of the integrable cases discussed we noticed that  $N_{fl}(x) = x^\alpha \phi(x)$ , where  $\alpha$  is some appropriate power that describes the overall growth of the fluctuations  $N_{fl}(x)$  of the spectral staircase as  $x \rightarrow \infty$ , and  $\phi(x)$  is a sum over the (primitive) lengths of classical periodic orbits and their multiples; one in particular finds that  $\mathbb{E}_x[N_{fl}^2] \sim \text{const.} x^{2\alpha}$ ,  $x \rightarrow \infty$ . The periodic orbit sum for  $\phi(x)$  was found to share some controllable convergence properties; it either was a Fourier series or, more typically, an almost periodic function whose representation by a trigonometric series converges in some Besicovitch norm. Now focussing on the chaotic case, we realize that a representation of  $N_{fl}(x)$  in terms of a sum over (primitive) periodic orbits, see for example (1.63), results in a considerably worse convergence. Due to the exponential proliferation (1.64) of the number of periodic orbits, a Gaussian smoothing was required in (1.63) in order to achieve a finite sum. Moreover, no power of  $x$  or  $\log x$  drops out explicitly in front of the periodic orbit term that could directly indicate the overall increase of  $N_{fl}(x)$  as  $x \rightarrow \infty$ . However, due to (2.56)  $\mathbb{E}_x[N_{fl}^2]$  is not bounded but rather grows logarithmically. Hence the overall increase of  $N_{fl}(x)$  is implicitly contained in the periodic orbit term itself and has to be uncovered by careful estimates of the sum. This remark should indicate the non-trivial problems that go along with a quantitative analysis of the distribution of  $W(y)$  for classically chaotic systems.

A strong hint at what has to be expected in the chaotic case is provided by a comparison with the distribution of  $W(y)$  for the non-trivial zeros of the Riemann zeta function. At the end of section 1.3 we pointed at the close analogy between the Selberg zeta function with its spectral zeros on the one hand, and the Riemann zeta function with its non-trivial zeros on the other hand. The asymptotics (1.86) of the counting function for the Riemann zeros now suggests to introduce the defolded variable  $x = \frac{t}{2\pi} \log \frac{t}{2\pi}$ . A ‘semiclassical’ analysis in the spirit of section 2.3 then reveals that  $l_H = \text{const.} \log x$  has to be chosen so that also  $L_{max} = \text{const.} \log x$ . According to (2.54) one therefore concludes that  $\mathbb{E}_x[N_{fl}^2] \sim \frac{1}{2\pi^2} \log \log x$ ,  $x \rightarrow \infty$ . In order to arrive at this result one has to take into account that lengths of periodic orbits correspond to logarithms of primes. Since the latter obviously have

multiplicity one, the semiclassical analysis requires to choose  $\overline{g} = 1$ , as in the case of classically chaotic systems without time-reversal invariance. The latter point is in accordance with the conjectured -and numerically confirmed- GUE behaviour of the local statistics of the Riemann zeros [63]. Already in 1946 Selberg proved [64]

$$\mathbb{E}_x \left[ N_{fl}^{2k} \right] = \frac{(2k)!}{k!} \left( \frac{1}{2\pi^2} \log \log x \right)^k + O \left( (\log \log x)^{k-\frac{1}{2}} \right), \quad (3.32)$$

so that the even moments of

$$W(y) = \frac{N_{fl}(y)}{\sqrt{\frac{1}{2\pi^2} \log \log x}}, \quad y \in I_x, \quad (3.33)$$

converge to the even moments of a standard Gaussian as  $x \rightarrow \infty$ . It was later noticed that an application of Selberg's method to the odd moments of  $W(y)$  shows that these asymptotically vanish, see for example [65]. The distribution of  $W(y)$ ,  $y \in I_x$ , therefore weakly converges to a Gaussian of zero mean and unit variance as  $x \rightarrow \infty$ . Selberg [66] extended this result to the corresponding limit distributions for a whole class of number theoretical zeta functions with functional equations. Since the zeta functions covered by Selberg's theorem are required to possess representations as ordinary Dirichlet series, the case of the Selberg zeta function  $Z(s)$  is not included. But based on the analogy between the zeros of the Riemann zeta function and quantum energies of a classically chaotic system as well as on numerical evidence, in [67] the following conjecture was established: If the classical dynamics are chaotic, with only isolated and unstable periodic orbits, and a scaling behaviour as in (1.40) holds, the limit distribution of  $W(y)$  is a standard Gaussian. In analogy to (3.31) it hence is conjectured that

$$\lim_{x \rightarrow \infty} \frac{1}{2\Delta x} \int_{x-\Delta x}^{x+\Delta x} g(W(y)) \rho \left( \frac{y-x}{2\Delta x} + \frac{1}{2} \right) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(w) e^{-\frac{1}{2}w^2} dw \quad (3.34)$$

for any bounded continuous function  $g(w)$  and any probability density  $\rho(\tau)$  on  $[0, 1]$ . The restriction of the conjecture to chaotic systems with scaling (1.40) stems from the analogy to the distribution of the zeros of zeta functions. Exactly in the scaling case one can directly encode the periodic orbit contribution to the spectral staircase in the complex phase of a semiclassical zeta function which is a function of the scaling variable  $E^\alpha$ , compare (1.40). Nevertheless, we expect the conjecture to hold also in more general situations, with the essential requirement that all periodic orbits be isolated and unstable. We also remark that the conjecture includes the case of arithmetic quantum chaos, where the local distribution of eigenvalues strongly deviates from the 'generic' situation for classically chaotic systems in that it rather shows a Poissonian behaviour than a GOE one [33, 34]. Concerning the global distribution of eigenvalues the differences between arithmetic and 'generic' systems only show up in the normalization of  $W(y)$  because in the arithmetic case one obtains  $\mathbb{E}_x[N_{fl}^2] \sim \frac{\sqrt{d}}{\pi} \frac{\sqrt{x}}{\log x}$ ,  $x \rightarrow \infty$ , see [29]. A preliminary version of the conjecture (3.34), which concerns  $W(y)$  evaluated only at the eigenvalues, is contained in [68].

In [69] the distribution of  $N_{fl}(y)$  has been computed numerically for two chaotic billiards in the unit disc (1.67) with the metric of constant negative curvature, one of them showing arithmetic quantum chaos, as well as for one integrable billiard. The normalization leading to the random variable  $W(y)$  was not taken into account. But since in numerical calculations the spectral interval  $I_x$  from which  $y$  is taken has to be kept finite, the non-existence of a limit as  $x \rightarrow \infty$  does not become apparent in numerical data. The numerical results of [69] show distributions that are somewhat closer to Gaussians in the two classically chaotic examples than in the classically integrable case. A detailed numerical analysis of the distribution of  $W(y)$  is presented in [70]. There six chaotic systems are compared with one pseudo-integrable and five integrable systems, and a clear evidence for Gaussian limit distributions to arise only in the classically chaotic cases is obtained.



The influence of periodic orbits on the limit distribution of  $W(y)$  entered our above discussion only indirectly through the representation of  $N_{fl}(x)$  by the complex phase of the Selberg zeta function or its semiclassical analogue. If one desires to uncover the role played by unstable periodic orbits in a more direct fashion one has to express  $N_{fl}(x)$  in terms of the Gutzwiller -or Selberg- trace formula. For this purpose let us again concentrate on the situation covered by the trace formula (1.48), or more specifically by the Selberg trace formula (1.71). Then  $N_{fl}(p)$ , where  $p$  denotes the appropriate momentum variable, can be represented by  $N_{\varepsilon, fl}(p)$  for  $\varepsilon \rightarrow 0$ , see (1.63). Alike in the semiclassical analysis of section 2.3 we now rewrite the sum over primitive periodic orbits  $\gamma_p$  as a sum over their respective lengths  $l_1 < l_2 < l_3 < \dots$ , thereby introducing the multiplicities  $g_n$  of  $l_n$ ,  $n = 1, 2, 3, \dots$ . That way  $N_{fl}(p)$  turns out to be a sum over the functions

$$a_n(t) := \frac{g_n}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \frac{e^{-i\frac{\pi}{2}k\tilde{\mu}_n}}{|\det(M_n^k - \mathbf{1})|^{\frac{1}{2}}} \sin(2\pi kt) \quad (3.35)$$

evaluated at  $t = \frac{p}{2\pi\hbar}l_n$ . From (3.35) we obtain that  $a_n(t)$  is a one-periodic continuous function with zero mean,

$$\int_0^1 a_n(t) dt = 0. \quad (3.36)$$

Moreover,

$$\sigma_n^2 := \int_0^1 |a_n(t)|^2 dt = \frac{g_n^2}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{|\det(M_n^k - \mathbf{1})|} \sim \frac{g_n^2}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-ku_n}, \quad n \rightarrow \infty. \quad (3.37)$$

If we now employ the periodic functions  $a_n(t)$  in order to express  $N_{fl}(p)$ , we have to consider  $\frac{p}{2\pi\hbar}l_n \bmod 1$ . In case the lengths  $l_n$  of primitive periodic orbits are linearly independent over  $\mathbb{Q}$ ,  $\frac{p}{2\pi\hbar}l_n \bmod 1$  becomes uniformly distributed on the interval  $[0, 1]$  as  $p \rightarrow \infty$  or  $\hbar \rightarrow 0$ . Furthermore, due to the result of [57] already alluded to above for integrable systems, the linear independence of  $l_{n_1}, \dots, l_{n_k}$  implies

$$\lim_{p \rightarrow \infty} \frac{1}{2\Delta p} \int_{p-\Delta p}^{p+\Delta p} a_{n_1}\left(\frac{p'}{2\pi\hbar}l_{n_1}\right) \dots a_{n_k}\left(\frac{p'}{2\pi\hbar}l_{n_k}\right) \frac{dp'}{2\pi\hbar} = \prod_{j=1}^k \int_0^1 a_{n_j}(t) dt, \quad (3.38)$$

where the interval  $[p - \Delta p, p + \Delta p]$  corresponds to the spectral interval  $I_x$  upon changing variables from the defolded energy variable  $y$  to the momentum variable  $p'$ . Thus  $N_{fl}(p')$  appears as a sum of the asymptotically independent random variables  $a_n(\frac{p'}{2\pi\hbar}l_n)$ .

When comparing the present situation with the classically integrable case one first of all notices that the required linear independence of the primitive lengths is not known to hold in a single classically chaotic example. Secondly, and more importantly, the sum

$$\sum_{n=1}^{\infty} \sigma_n^2 = \sum_{n=1}^{\infty} \int_0^1 |a_n(t)|^2 dt \quad (3.39)$$

can be shown to diverge. To this end we remark that obviously the leading behaviour of  $\sigma_n^2$  as  $n \rightarrow \infty$  is determined by the contribution from  $k = 1$  in (3.37). If we furthermore recall the discussion of the r.h.s. of (2.53), we obtain that in case  $\bar{g} = 2$

$$\frac{1}{2\pi^2} \sum_{l_n \leq l} g_n^2 e^{-u_n} \approx \frac{\bar{g}}{2\pi^2} \sum_{\gamma_p, l_{\gamma_p} \leq l} e^{-u_{\gamma_p}} \sim \frac{1}{\pi^2} \log l, \quad l \rightarrow \infty. \quad (3.40)$$

Thus, if we consider the random variables  $a_n(\theta_n)$ , where the  $\theta_n$ 's are independent random variables uniformly distributed on the interval  $[0, 1]$ , their variances  $\sigma_n^2$  sum up to

$$B_N := \sum_{n=1}^N \sigma_n^2 \sim \frac{1}{\pi^2} \log l_N \sim \frac{1}{\pi^2} \log \left[ \frac{1}{\tau} \log(2N \log 2N) \right] \sim \frac{1}{\pi^2} \log \log N, \quad N \rightarrow \infty. \quad (3.41)$$

The asymptotic relation  $l_N \sim \frac{1}{\tau} \log(2N \log 2N)$ ,  $N \rightarrow \infty$ , leading to (3.41) immediately derives from (1.64). For the arithmetic systems with their exceptionally large multiplicities of lengths [49]  $B_N$  diverges more strongly than in the ‘generic’ case covered by (3.41). In the classically integrable situation we noticed that both conditions (3.29) ensured the existence of a limit distribution and allowed for conclusion on its properties. In the present situation the second condition is violated, which indicates bad convergence properties of the sum  $\sum_n a_n(t)$ . However, it is suggested in [59] that due to the divergence of (3.39) one should employ the Lindeberg-Feller version of the central limit theorem, see for example [71]. This states that the distribution of the random variable

$$\xi_N := \frac{\sum_{n=1}^N a_n(\theta_n)}{\sqrt{B_N}} \quad (3.42)$$

converges as  $N \rightarrow \infty$  to a Gaussian with zero mean and unit variance.

We now again adopt the point of view, already employed in section 2.3, that the periodic orbit sum for  $N_{fl}(p)$  is dominated by the terms with lengths  $l_n \leq l_H = 4\pi p \hbar \bar{d}$ . If we then denote the index corresponding to  $l_H$  by  $N_H$ ,  $l_{N_H} = l_H$ , we first of all notice that  $N_{fl}(p)$  is essentially given by

$$\sum_{n=1}^{N_H} a_n \left( \frac{p}{2\pi \hbar} l_n \right) . \quad (3.43)$$

We now also cut off the sum of the variances  $\sigma_n^2$  that refer to the individual terms in (3.43) at  $N_H$ , and hence obtain

$$B_{N_H} = \sum_{n=1}^{N_H} \sigma_n^2 \sim \frac{1}{\pi^2} \log l_H \sim \frac{1}{2} \Sigma_\infty^2(E) \sim \mathbb{E}_x [N_{fl}^2] , \quad (3.44)$$

see (3.41), (2.54), and (2.56). The relation (3.44) is remarkable since it states that in the semiclassical limit the true variance of  $N_{fl}(p)$ , which appears on the very r.h.s., is asymptotic to the sum of the variances  $\sigma_n^2$  when this is cut off at  $N_H$ . Therefore the two assumptions: statistical independence of the  $a_n$ ’s and truncation of the periodic orbit sum at  $l_H$ , merge into a coherent picture. We conclude that thus  $\xi_{N_H}$  essentially yields  $W(y)$ ,

$$W(y) = \frac{N_{fl}(y)}{\sqrt{\mathbb{E}_x [N_{fl}^2]}} \sim \xi_{N_H} = \frac{\sum_{n=1}^{N_H} a_n(\frac{p}{2\pi \hbar} l_n)}{\sqrt{B_{N_H}}} , \quad (3.45)$$

and that the limit distributions coincide. Here  $y \in I_x$ , and  $x$  corresponds to  $l_H$  via  $l_H = 4\pi \hbar \sqrt{\bar{d}x}$ . Notice that the semiclassical limit  $l_H \rightarrow \infty$  has to be performed in (3.45) as  $x \rightarrow \infty$  on the l.h.s., and as  $N_H \rightarrow \infty$  on the r.h.s. We therefore expect the limit distribution of  $\xi_N$  as  $N \rightarrow \infty$ , see (3.42), to reproduce the limit distribution of  $W(y)$ ,  $y \in I_x$ , as  $x \rightarrow \infty$ . At this point we want to stress that the above considerations do not constitute a proof of (3.34), since the assumptions made at various stages remain to be verified. However, we believe that all evidence is in favour of the conjecture.

The above discussion of the limit distribution of  $W(y)$  in the classically chaotic case was restricted to systems with only isolated and unstable periodic orbits. Only for those the zeta function analogy works and the periodic orbit expression for  $N_{fl}(x)$  has the form required by (3.35). In order to illustrate the necessity of the condition on the periodic orbits let us discuss the stadium billiard in some detail. The classical dynamics are known to show the K-property [72], and hence the stadium billiard qualifies by all means as a chaotic system in the usual sense. However, not all periodic orbits are isolated and unstable because of the presence of the bouncing ball orbits that form a one-parameter family of periodic orbits. In addition to the contribution of the isolated and unstable orbits the Gutzwiller trace formula for the stadium billiard therefore contains a different term caused by the bouncing ball orbits. Thus the spectral staircase expressed in the momentum variable reads

$$N(p) = \overline{N}(p) + N_{fl,u}(p) + N_{bb}(p) , \quad (3.46)$$

where  $N_{fl,u}(p)$  is the usual contribution of the unstable periodic orbits. In [54] an explicit expression for the contribution of the bouncing ball orbits is derived,

$$N_{bb}(p) = \frac{b}{2\sqrt{\pi^3 a}} p^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \cos\left(2anp - \frac{3\pi}{4}\right) + O\left(p^{-\frac{1}{2}}\right). \quad (3.47)$$

Here  $b$  denotes the length of the two parallel edges of the billiard domain, and  $a$  is the radius of the two half-circles. In fact, (3.47) corresponds to the desymmetrized stadium billiard whose domain consists of a quarter of the stadium domain, see [54] for details. In terms of the defolded energy variable  $x = \bar{d}p^2$  the variances of the two periodic orbit contributions in (3.46) read

$$\mathbb{E}_x [N_{fl,u}^2] \sim \frac{1}{2\pi^2} \log x \quad \text{and} \quad \mathbb{E}_x [N_{bb}^2] \sim \frac{b^2 \zeta(3)}{8\pi^3 a \sqrt{\bar{d}}} x^{\frac{1}{2}}, \quad x \rightarrow \infty. \quad (3.48)$$

The first term is just (2.56), whereas the second term was determined in [54]. Hence

$$\mathbb{E}_x [(N_{fl,u} + N_{bb})^2] \sim \frac{b^2 \zeta(3)}{8\pi^3 a \sqrt{\bar{d}}} x^{\frac{1}{2}}, \quad x \rightarrow \infty, \quad (3.49)$$

so that  $W(x)$  is asymptotically completely determined by the bouncing ball contribution

$$W_{bb}(x) := \frac{N_{bb}(x)}{\sqrt{\mathbb{E}_x [N_{bb}^2]}} \sim \sqrt{\frac{2}{\zeta(3)}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \cos\left(2an\sqrt{\frac{x}{\bar{d}}} - \frac{3\pi}{4}\right). \quad (3.50)$$

Thus  $W_{bb}(x)$ , considered as a function of  $\sqrt{x}$ , is asymptotic to a continuous periodic function with Fourier series given by (3.50). Since  $\sum_n n^{-\frac{3}{2}} < \infty$  we can moreover estimate

$$|W_{bb}(x)| \leq \sqrt{\frac{2}{\zeta(3)}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} = \sqrt{\frac{2}{\zeta(3)}} \zeta\left(\frac{3}{2}\right). \quad (3.51)$$

This finite bound implies that the density  $p(w)$  of the limit distribution of  $W(y)$  for the complete desymmetrized stadium billiard vanishes outside the finite interval  $[-c, +c]$ , where  $c$  denotes the constant on the r.h.s. of (3.51). The limit distribution is therefore clearly not Gaussian, and this violation of (3.34) is solely caused by the bouncing ball orbits. We remark that this observation is not restricted to the stadium billiard, but will occur in all systems with families of periodic orbits. These always yield variances of their contributions to the spectral staircase that exceed the logarithmic contribution (2.56) of the unstable orbits. By dividing through the rms fluctuations of the spectral staircase in the definition of  $W(y)$ , the influence of the otherwise dominating unstable orbits hence vanishes asymptotically.

### 3.4 Summary of the Global Statistics

After having discussed various aspects of the global distribution of eigenvalues for classically integrable as well as chaotic systems in some detail we now want to summarize the emerging perception of the global statistics. But before we again stress some points that we already remarked before at several places.

1. Statements about local as well as global eigenvalue statistics always refer to ‘generic’ systems in a given class. A precise definition of what this means has to be given in each particular case. A typical example would be some statement concerning eigenvalue statistics on tori, see (2.26), where one would have to specify the set of parameters  $A, B, C$  to which the statement should apply. It seems to be almost impossible, however, to give a general characterization of a ‘generic’ classically integrable (or chaotic) system in a unified manner.

2. Concerning the local statistics of eigenvalues it is assumed that it can ‘generically’ be described by the appropriate random matrix ensembles for classically chaotic systems [41], and by Poissonian statistics for classically integrable systems [42].
3. We furthermore assume that the description via random matrix ensembles or Poisson statistics extends to hold in the limit  $L \rightarrow \infty$  once this is performed in the universality regime  $L/L_{max} \rightarrow 0$ . We base this assumption on the following observation: The local statistics of eigenvalues emerges as the limit of  $x \rightarrow \infty$  of, say,  $E_x(k; L)$  for finite  $L$ . Passing to  $L \rightarrow \infty$  within the universality regime appears to leave the analogy to, say, the GOE in the number variance unchanged, see (2.51).

In the course of our discussion of eigenvalue distributions it became clear that the random variable  $\eta_L(y)$ ,  $y \in I_x$ , appears to be most suited for a unified treatment of spectral statistics on all scales. After the limit  $x \rightarrow \infty$  has been performed, the distribution of  $\eta_L$  with finite  $L$  yields the local statistics. According to (3.8) and the assumption stated in 2., then

$$\nu_{\eta_L}(d\eta) = \sqrt{\Sigma^2(L)} E\left(\eta\sqrt{\Sigma^2(L)} + L; L\right) d\eta, \quad (3.52)$$

where  $E(k; L)$  is either given by random matrix theory or by Poisson statistics.

The transition to the global scale now proceeds under the assumption in 3. We hence conclude that the distribution of  $\eta_L(y)$ ,  $y \in I_x$ , converges for  $x \rightarrow \infty$ ,  $L \rightarrow \infty$ ,  $L/L_{max} \rightarrow 0$  to the same limit distribution as the respective distributions of  $\eta_L$  in the random matrix ensembles or for a Poisson random process when  $L \rightarrow \infty$ . Since the latter limit distributions are standard Gaussians, see [53] and (3.11), we therefore expect the central limit theorem (3.13) to hold for all ‘generic’ systems, independent of their respective classical limits. It thus appears that the global spectral statistics in the universality regime are not sensitive enough a measure of eigenvalue correlations in order to yield information about specific features of a quantum system.

Finally we have to consider the limit  $x \rightarrow \infty$ ,  $L \rightarrow \infty$ ,  $L/L_{max} \rightarrow \infty$ , i.e., the transition to the global scale in the saturation regime. In section 3.1 we argued that the limit distributions of  $\eta_L(y)$ ,  $y \in I_x$ , then coincide with the limit distributions of  $W(y)$ ,  $y \in I_x$ , as  $x \rightarrow \infty$ . This conclusion was drawn from the apparent asymptotic independence of the random variables  $N_{fl}(y)$ ,  $y \in I_x$ , and  $N_{fl}(y)$ ,  $y \in I_{x+L}$ , as  $L/L_{max} \rightarrow \infty$ , which was discussed at the end of section 2.3. The principal observation then made in sections 3.2 and 3.3 was that the limit distributions of  $W(y)$  allow to distinguish classically integrable systems from classically chaotic ones with only isolated and unstable periodic orbits. Only the latter yield Gaussian limit distributions, whereas in all other cases the distributions of  $W(y)$  are found to converge to a non-Gaussian limit.

We are now in a position to summarize the above findings in the following generalization of the conjecture (3.34) introduced in [67]. Considering the limit distribution of the random variable  $\eta_L(y)$ ,  $y \in I_x$ , we claim that

$$\lim_{\substack{x \rightarrow \infty \\ L \rightarrow \infty}} \int_{-\infty}^{+\infty} g(\eta) \nu_{\eta_L, x}(d\eta) = \int_{-\infty}^{+\infty} g(\eta) P(\eta) d\eta, \quad (3.53)$$

for any bounded continuous function  $g(\eta)$ . The limit distribution  $P(\eta) d\eta$  shall have a density  $P(\eta)$  that depends on the way the double limit is performed, as well as on the type of the classical dynamics.

1.  $L/L_{max} \rightarrow 0$ : For ‘generic’ systems, be they classically integrable or chaotic, one obtains

$$P(\eta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\eta^2}. \quad (3.54)$$

2.  $L/L_{max} \rightarrow \infty$ : The limit density  $P(\eta)$  depends on the type of the corresponding classical system. When the latter is integrable (or chaotic with families of periodic orbits)  $P(\eta)$  is non-Gaussian. Most probably it will decay faster than a Gaussian,

$$\lim_{\eta \rightarrow \pm\infty} P(\eta) e^{\frac{1}{2}\eta^2} = 0. \quad (3.55)$$

In certain cases it can even have a compact support. If the classical dynamics are chaotic, with only isolated and unstable periodic orbits, then a standard Gaussian as in (3.54) is to be observed.

In the field of quantum chaos so far mainly the local distribution of eigenvalues was investigated; in particular the distribution of nearest neighbour level spacings played an eminent role. The results as described by the random matrix/Poissonian conjecture, however, seem to be somewhat counter-intuitive. A Poissonian random process, which describes classically integrable, i.e. regular, systems, is characterized by the absence of any correlations. On the other hand, the behaviour of classically chaotic systems shows strong correlations. In this respect the situation concerning the global distribution of eigenvalues in the saturation regime seems to be more confirm to intuition. The Gaussian limit distribution emerging for  $W(y)$  in the chaotic case is the most random of all possible probability distributions that have a density and are of a fixed common variance. If we introduce the spectral entropy

$$\mathcal{E}[p] := - \int_{-\infty}^{+\infty} p(w) \log p(w) dw , \quad (3.56)$$

which provides a quantitative measure of a mean unlikelihood for  $W(y)$  to have a certain value, the distribution with maximal entropy characterizes the most random spectral fluctuations. It is well known that under the constraint of a fixed variance,  $\mathcal{E}[p]$  is maximized by a Gaussian of zero mean. Therefore, quantum spectra that have a Gaussian limit distribution for  $W(y)$  or  $\eta_L(y)$  possess, on a global scale, the most random fluctuations. This seems to be a convenient characterization of quantum chaos.

## Acknowledgement

I thank the organizers of the 3rd International Summer School/Conference *Let's face chaos through nonlinear dynamics* at the University of Maribor for their warm hospitality and for the opportunity to present these lectures.

## References

- [1] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai: *Ergodic Theory*. Springer-Verlag, New York-Heidelberg-Berlin 1982.
- [2] M.C. Gutzwiller: *Chaos in Classical and Quantum Mechanics*. Springer-Verlag, New York-Berlin-Heidelberg 1990.
- [3] M.-J. Giannoni, A. Voros, J. Zinn-Justin (eds.): *Chaos and quantum physics*. Les Houches 1989, Session LII. North Holland, Amsterdam-London-New York-Tokyo 1991.
- [4] V.I. Arnold: *Mathematical Methods of Classical Mechanics*. Second edition, Springer-Verlag, New York-Berlin-Heidelberg 1989.
- [5] D. Robert: *Autour de l'Approximation Semi-Classique*. Birkhäuser, Boston 1987.
- [6] A. Grigis, J. Sjöstrand: *Microlocal Analysis for Differential Operators*. London Mathematical Society Lecture Note Series Vol. 196, Cambridge University Press, Cambridge 1994.
- [7] W. Pauli: *Ausgewählte Kapitel aus der Feldquantisierung*. Boringheri, Turin 1952.
- [8] M.C. Gutzwiller: J. Math. Phys. **8** (1967) 1979.
- [9] M.C. Gutzwiller: J. Math. Phys. **12** (1971) 343.
- [10] M. Sieber, F. Steiner: Phys. Lett. **A144** (1990) 159.

- [11] D. Ruelle: *Thermodynamic Formalism*. Encyclopedia of Mathematics and its Applications Vol. 5, Addison-Wesley, Reading, Massachusetts 1978.
- [12] P. Walters: *An Introduction to Ergodic Theory*. Springer-Verlag, New York-Heidelberg-Berlin 1982.
- [13] J.J. Duistermaat, V. Guillemin: *Invent. Math.* **29** (1975) 39.
- [14] T. Paul, A. Uribe: *C.R. Acad. Sci. Paris Série I* **313** (1991) 217.
- [15] E. Meinrenken: *Rep. Math. Phys.* **31** (1992) 279.
- [16] H.P. McKean, I.M. Singer: *J. Diff. Geom.* **1** (1967) 43.
- [17] R. Aurich, M. Sieber, F. Steiner: *Phys. Rev. Lett.* **61** (1988) 483.
- [18] R. Aurich, C. Matthies, M. Sieber, F. Steiner: *Phys. Rev. Lett.* **68** (1992) 1629.
- [19] R. Aurich, J. Bolte: *Mod. Phys. Lett.* **B6** (1992) 1691.
- [20] D.A. Hejhal: *The Selberg Trace Formula for  $PSL(2, \mathbb{R})$  Vol. I*. Lecture Notes in Mathematics 548, Springer-Verlag, Berlin-Heidelberg-New York 1976.
- [21] J.G. Ratcliffe: *Foundations of Hyperbolic Manifolds*. Springer-Verlag, New York-Berlin-Heidelberg 1994.
- [22] N.L. Balazs, A. Voros: *Phys. Rep.* **143** (1986) 109.
- [23] A. Selberg: *J. Indian Math. Soc.* **20** (1956) 47.
- [24] F. Steiner: *Phys. Lett.* **188B** (1987) 447.
- [25] M. Sieber, F. Steiner: *Phys. Rev. Lett.* **67** (1991) 1941.
- [26] G. Tanner, P. Scherer, E.B. Bogomolny, B. Eckhardt, D. Wintgen: *Phys. Rev. Lett.* **67** (1991) 2410.
- [27] E.C. Titchmarsh: *The Theory of the Riemann Zeta-function*. Second edition revised by D.R. Heath-Brown, Clarendon Press, Oxford 1986.
- [28] P.M. Bleher, Z. Cheng, F.J. Dyson, J.L. Lebowitz: *Commun. Math. Phys.* **154** (1993) 433.
- [29] J. Bolte: *Int. J. Mod. Phys.* **B7** (1993) 4451.
- [30] M.V. Berry: *Nonlinearity* **1** (1988) 399.
- [31] R. Aurich, F. Steiner: *Physica* **D82** (1995) 266.
- [32] R. Aurich, F. Steiner: *Physica* **D43** (1990) 155.
- [33] E.B. Bogomolny, B. Georgeot, M.-J. Giannoni, C. Schmit: *Phys. Rev. Lett.* **69** (1992) 1477.
- [34] J. Bolte, G. Steil, F. Steiner: *Phys. Rev. Lett.* **69** (1992) 2188.
- [35] W. Luo, P. Sarnak: *Commun. Math. Phys.* **161** (1994) 419.
- [36] P. Sarnak: *Israel Mathematical Conference Proceedings* **8** (1995) 183.
- [37] N.M. Katz, P. Sarnak: *The spacing distribution between zeros of zeta functions*. Preprint, Princeton University 1996.
- [38] M.L. Mehta: *Random Matrices*. Revised and enlarged second edition. Academic Press, San Diego 1991.
- [39] O. Costin, J.L. Lebowitz: *Phys. Rev. Lett.* **75** (1995) 69.
- [40] C.E. Porter (ed.): *Statistical Theories of Spectra: Fluctuations*. Academic Press, New York and London 1965.

- [41] O. Bohigas, M.-J. Giannoni, C. Schmit: Phys. Rev. Lett. **52** (1984) 1.
- [42] M.V. Berry, M. Tabor: Proc. R. Soc. Lond. **A356** (1977) 375.
- [43] Z. Cheng, J.L. Lebowitz: Phys. Rev. **A44** (1991) R3399.
- [44] P. Sarnak: *Values at integers of binary quadratic forms*. Preprint, Princeton University 1996.
- [45] P.M. Bleher, J.L. Lebowitz: Ann. Inst. H. Poincaré **31** (1995) 27.
- [46] M.V. Berry: Proc. R. Soc. Lond. **A400** (1985) 229.
- [47] W. Parry, M. Pollicott: *Zeta functions and the periodic orbit structure of hyperbolic dynamics*. Astérisque **187-188**, Société Mathématique de France 1990.
- [48] J.H. Hannay, A.M. Ozorio De Almeida: J. Phys. A: Math. Gen. **17** (1984) 3429.
- [49] J. Bolte: Nonlinearity **6** (1993) 935.
- [50] S. Johansson: *Traces in Arithmetic Fuchsian Groups*. Preprint Dept. Math. Chalmers Univ. of Techn. and Göteborg Univ. 1996.
- [51] E.B. Bogomolny, J.P. Keating: Phys. Rev. Lett. **77** (1996) 1472.
- [52] A. Bäcker, P. Stifter, F. Steiner: Phys. Rev. **E 52** (1995) 2463-2472.
- [53] P.M. Bleher, J.L. Lebowitz: J. Stat. Phys. **74** (1994) 167.
- [54] M. Sieber, U. Smilansky, S.C. Creagh, R.G. Littlejohn: J. Phys. A: Math. Gen. **26** (1993) 6217.
- [55] P.M. Bleher: Duke Math. J. **74** (1994) 45.
- [56] D.V. Kosygin, A.A. Minasov, Ya.G. Sinai: Uspekhi Mat. Nauk **48** (1993) 3.
- [57] D.R. Heath-Brown: Acta Arithmetica **60** (1992) 389.
- [58] P.M. Bleher: Duke Math. J. **67** (1992) 461.
- [59] P.M. Bleher: *Trace Formula for Quantum Integrable Systems, Lattice-Point Problem, and Small Divisors*. Preprint Indiana Univ.-Purdue Univ. 96-11, Indianapolis 1996.
- [60] P.M. Bleher: Duke Math. J. **70** (1993) 655.
- [61] P.M. Bleher, D.V. Kosygin, Ya.G. Sinai: Commun. Math. Phys. **170** (1995) 375.
- [62] R. Schubert: *The Trace Formula and the Distribution of Eigenvalues of Schrödinger Operators on Manifolds all of whose Geodesics are Closed*. DESY-report, DESY 95-090, 1995.
- [63] A.M. Odlyzko: Math. Comp. **48** (1987) 273.
- [64] A. Selberg: Archiv for Matematik og Naturvidenskab **B48** (1946) 89.
- [65] H.L. Montgomery, in: K.E. Aubert, E. Bombieri, D. Goldfeld (eds.): *Number Theory, Trace Formulas and Discrete Groups*. Academic Press, New York 1989.
- [66] A. Selberg: *Collected Papers, Vol. II*. Springer-Verlag, Berlin-Heidelberg-New York 1991.
- [67] R. Aurich, J. Bolte, F. Steiner: Phys. Rev. Lett. **73** (1994) 1356.
- [68] F. Steiner, in: R. Ansorge (ed.): *Schlaglichter der Forschung. Zum 75. Jahrestag der Universität Hamburg*. Dietrich Reimer Verlag, Berlin-Hamburg 1994.
- [69] E.B. Bogomolny, C. Schmit: Nonlinearity **6** (1993) 523.

- [70] R. Aurich, A. Bäcker, F. Steiner: *Mode fluctuations as fingerprints of chaotic and non-chaotic systems*. Preprint ULM-TP/96-2, 1996. To appear in Int. J. Mod. Phys. B.
- [71] V.V. Petrov: *Limit Theorems of Probability Theory*. Clarendon Press, Oxford 1995.
- [72] L.A. Bunimovich: Commun. Math. Phys. **65** (1979) 295.